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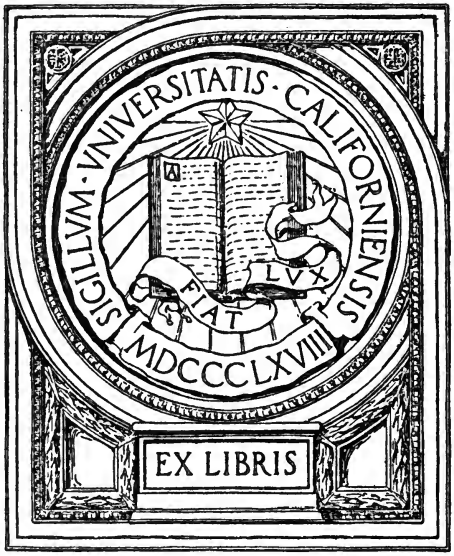

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SCHOOL ALGEBRA.

BY

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MADISON, WIS.:
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PREFACE.

The present volume is issued in the hope that it will assist the teacher in the effort to get pupils to *think* and that it will induce pupils to place the subject of Algebra on a rational instead of an arbitrary basis, to work from principles rather than from rules.

In the first part of the work we have pursued what is termed the *inductive* method, but we do not wish this understood as that method which infers general principles from an accumulation of particular cases. This is the inductive reasoning of the natural sciences, but we believe it is never legitimate in mathematics. Induction, as we use the term, means that method which proceeds from the particular to the general. By particular cases, which gradually increase in generality, the mind of the learner is prepared to appreciate the general case, but this general case must so present itself to the learner's mind that he sees that the truth stated *must* be so and cannot *possibly* be otherwise.

It will be noticed that we have not thought it necessary to complete one subject before taking up another, but subjects have sometimes been treated in an elementary way at first and more completely at some subsequent part of the book. We believe that by this plan students can follow the work more easily and with more profit, and at the same time we are enabled to treat some subjects, especially Factors, Multiples, and Fractions, more fully than is ordinarily done.

We have placed the principles governing the use of parentheses before the four fundamental operations of addition, subtraction, multiplication, and division, and have made the latter depend upon the former. This we think enables us to treat the four fundamental operations in a way which is more rational to beginners than is given when the usual order is pursued.

The subject of equations is early introduced, and is distributed through the book instead of being given together in one place. This keeps up the interest in the subject and prevents the student from getting the idea that he is learning a mass of theory which has no practical application.

Indices and Surds appear after Quadratics for the reason that these subjects are more difficult than Quadratics.

Additive and subtractive terms are distinguished from positive and negative quantities, and the latter are postponed until after the four fundamental operations.

This work contains about 3000 examples besides several hundred problems and inductive exercises. Many of these are original and many are taken from the German and French collections and from the English examination papers. The answers are not printed in the book for a reason that every teacher of Algebra can readily assign, but the answers are issued in pamphlet form for the use of teachers only.

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December, 1890.

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SCHOOL ALGEBRA.

CHAPTER I.

FIRST PRINCIPLES.

EXERCISE 1.

ILLUSTRATING HOW A LETTER MAY BE USED TO REPRESENT A NUMBER.

1. In Algebra, as in Arithmetic, the symbol for **Plus** is + and the symbol for **Minus** is −.

1. How many dozen are 6 dozen + 4 dozen + 2 dozen?

2. How many score are 6 score + 4 score + 2 score?

3. How many hundred are 6 hundred + 4 hundred + 2 hundred?

4. How many times 100 are 6 times 100 + 4 times 100 + 2 times 100?

5. How many times 10 are 6 times 10 + 4 times 10 + 2 times 10?

6. How many times 7 are 6 times 7 + 4 times 7 + 2 times 7?

7. Six times *any number* plus four times the *same number* plus two times the *same number* are how many times *that number?*

2. *In Algebra letters are often used to represent or stand for numbers.*

3. In Algebra, as in Arithmetic, the symbol for times is $\times$.

In any statement like $6\times100+3\times100-5\times100$, *i. e.*, in any statement where addition, or subtraction, and multiplication occur together, the multiplications must always be performed before any addition or subtraction takes place. Thus, $6\times100+3$ does not mean 6×103, but means $600+3$.

8. If t stands for 10, how many times 10 are $4\times t+7\times t-8\times t$?

9. If t stands for 2, how many times 2 are $4\times t+7\times t-8\times t$?

10. If t stands for 3, how many times 3 are $4\times t+7\times t-8\times t$?

11. If s stands for 6, how many times 6 are $7\times s+5\times s-3\times s$?

12. If s stands for 7, how many times 7 are $7\times s+5\times s-3\times s$?

13. Seven times *a certain number* plus five times the *same number* minus three times the *same number* are how many times *that number?*

14. If n stands for *a certain number*, how many times that number are $7\times n+5\times n-3\times n$?

15. In question 14 can n stand for 17? for 100? for 25? for $\frac{1}{2}$? for $\frac{2}{3}$? for $1\frac{1}{2}$?

In question 14, n can stand for ANY NUMBER WHATEVER, *provided it stands for the same number throughout question and answer.*

16. If a stands for a certain number, $8\times a+4\times a-5\times a$ are how many times that number?

17. If b stands for a certain number, $8\times b+4\times b-5\times b$ are how many times that number?

18. Could some other letter than a, b, n, s, or t be used to represent a number?

4. The preceding questions suggest the following principle:

Any letter may be used to represent or stand for any number, provided that the same letter represents the same number throughout the same question and answer.

5. In Algebra it is usual to omit the sign $\times$ in a product like $7 \times n$ and write merely $7n$. It is then read "seven n," instead of "seven *times* n," but of course it always *means* seven times n; and this meaning the learner must keep in mind. Thus, if b stands for $31\frac{1}{2}$, then $7b$ stands for 7 times $31\frac{1}{2}$.

Evidently when the sign $\times$ occurs between *figures* it cannot be omitted. Thus we can write $7b$ for $7 \times b$, but we cannot write $731\frac{1}{2}$ for $7 \times 31\frac{1}{2}$, for $731\frac{1}{2}$ has a different meaning already given to it.

6. The number written before a letter to show how many times the number represented by the letter is taken, is called the **Coefficient** * of the letter. Thus, in $7b$, 7 is called the coefficient of b.

19. What does $5b$ mean? How much is this if b equals 10? What is the 5 called?

20. What does $10n$ mean? How much is this if n equals 8? What is the 10 called?

EXERCISE 2.

LEADING TO THE IDEA OF AN ALGEBRAIC EXPRESSION.

1. What does $7a-3a+5a$ equal, if a stands for 7?

2. What does $10a+4a-9a$ equal, if a stands for 6?

---

* A more general definition of coefficient is given in Art. 14.

3. What does $35a-7a-8a$ equal, if a stands for $\frac{2}{10}$?

4. What does $9a+6a-7a-4a+2a$ equal, if a stands for $\frac{1}{12}$?

5. What does $25n+\frac{2}{3}$ equal, if n stands for $\frac{1}{3}$?

6. What does $25n-6n-\frac{1}{4}$ equal, if n stands for $\frac{1}{4}$?

7. Anything, whether short and simple or long and complicated, which is or may be considered to be equal to some number, is called an **Expression.**

The number to which the expression is equal is called the **Value of the Expression.**

The number which a letter stands for is called the **Value of the Letter.**

7. What is the value of the expression $10c+2c-5c-6c+3c$, if the value of c is 4?

8. What is the value of the expression $12m-5m-6m+3m$, if the value of m is 12?

9. What is the value of the expression $7a-4a-3a+a$, if the value of a is 2?

10. What is the value of the expression $16d+24d-10d+25$, if the value of d is $\frac{1}{10}$?

EXERCISE 3.

LEADING TO THE NOTION OF AN ALGEBRAIC EQUATION AND THE DISTINCTION BETWEEN KNOWN AND UNKNOWN NUMBERS.

8. In Algebra, as in Arithmetic, the symbol for Equals is $=$.

1. What is the value of $12a$, if $a=2$? if $a=5$? if $a=7$? if $a=\frac{1}{2}$? if $a=6\frac{1}{3}$?

2. What must a equal if the value of $12a$ is 36? if the value of $12a$ is 48? if the value of $12a$ is 72? if the value of $12a$ is 8? if the value of $12a$ is 20?

3. If $x=8$, what does $7x$ equal? If $7x=56$, what does x equal? If $7x=49$, what does x equal?

4. If $x=9$, what does $11x$ equal? If $11x=99$, what does x equal? If $11x=121$, what does x equal?

5. If $a=12$, what does $3a+5a$ equal? If $3a+5a=96$, what does a equal? If $3a+5a=4$, what does a equal?

6. If $n=6$, what does $6n+5n+n$ equal? If $6n+5n+n=72$, what does n equal? If $6n+5n+n=108$, what does n equal?

7. If $7b-6b+3b=20$, what is the value of b?

9. The statement of equality which exists between two expressions is called an **Equation**, and the parts on either side of the sign = are called the **Members** of the equation. The expression on the left-hand side of the sign = is called the **Left** or **First Member**, and the expression on the right-hand side of the sign = is called the **Right** or **Second Member.**

8. If $3k+2k+k=120$, what is the value of k?

9. If $4x+5x-3x=3$, what is the value of x?

10. If $7x-2x+x=6$, what is the value of x?

11. If $2y=2\frac{1}{2}$, what is the value of y?

12. If $3z+2z=12$, what is the valve of z?

13. If $c=\frac{3}{4}$, what is the value of $12c+\frac{1}{4}-9c$?

14. If $d=1\frac{1}{2}$, what is the value of $8d+5d-10$?

15. If $f=1.2$, what is the value of $20f-7f-3f$?

16. If $100w-78w+15w=74$, what is the value of w?

17. If $17u-5u-4u=66$, what is the value of u?

18. If $g=5$, what is the value of $25g-5g+5$?

19. If $h=100$, what is the value of $20h-6h-3h+100$?

20. If $18v+11v+2v+9v=80$, what is the value of v?

10. A careful inspection of the questions of this exercise shows that when there is only one letter in an expression, two cases may arise: first, the value of the letter may be given and the value of the expression required; second, the value of the expression may be given and the value of the letter required.

In the first case the value of the letter is *known* or *given*, and in the second case the value of the letter is *unknown* or *required*.

Thus we see that in Algebra there are two kinds of numbers, called respectively **Known** and **Unknown**, either of which may be represented by a letter; and as it is possible that both kinds of numbers will appear in the same question, it is customary to distinguish between them by representing the known numbers by the first and intermediate letters of the alphabet, and the unknown numbers by the last letters of the alphabet.

EXERCISE 4.

SYMBOLIC EXPRESSIONS.

1. A coat and hat cost \$24; the hat cost \$4. What does \$24—\$4 stand for?

2. A coat and hat cost \$24; the hat cost \$$x$. What does \$24—\$$x$ stand for?

3. A coat and hat cost \$24; the hat cost \$$x$ and the coat 5 times as much as the hat. What does \$5x stand for?

What does \$24—\$5x stand for?

What does \$$x$+\$5x, or \$6x, stand for?

4. If n stands for a certain number, what expression will stand for a number which is 10 larger?

5. If a stands for a certain number, what will stand for a number which is 25 smaller?

6. There are two numbers; the second number is five more than twice the first number. If n represents the first number, what expression will represent the second number?

7. Of two numbers the second is 12 less than 5 times the first. If n stands for the first number, what will stand for the second number?

8. How many feet in n yards?

9. How many feet in n yards plus 5 yards?

10. How many feet in n yards plus 5 feet?

11. A room is a yards wide and twice as long as it is wide. How many yards long is the room? How many feet long is the room?

12. In x years a man will be 36 years old. What is his present age?

13. A man is now 40 years old. How old will he be a years from now? How old will he be $2a$ years from now? How old was he $3b$ years ago?

EXERCISE 5.

PROBLEMS.

1. A coat and hat cost \$24. The coat cost 5 times as much as the hat. What was the cost of each?

SOLUTION BY ARITHMETIC.

Five times *the cost of the hat* = the cost of the coat.
Once *the cost of the hat* = the cost of the hat.
Therefore, six times *the cost of the hat* = the cost of coat and hat.
But the cost of the coat and hat is \$24.
Hence, six times *the cost of the hat* = \$24.
Therefore, *the cost of the hat* = $\frac{1}{6}$ of \$24, or \$4.
Consequently the cost of the coat was \$20.

SOLUTION BY ALGEBRA.

The cost of the hat is a certain number of dollars, and that number, whatever it is, we may represent by a letter. For example, we may say,

Let	x = number of dollars the hat cost.
Then	$5x$ = number of dollars the coat cost.
Hence, $5x+x$, or $6x$,	= number of dollars that both hat and coat cost.
But	24 = number of dollars that both hat and coat cost.
Therefore	$6x = 24$.
Then	$x = 4$,
and	$5x = 20$.

Therefore the hat cost \$4, and the coat cost \$20.

The student should compare very carefully the two solutions of this problem above given. It will be noticed that the arithmetic and algebraic solutions are not so different from each other as would appear at first sight. In fact, to change the first solution to the second nothing need be done except to replace the words "*the cost of the hat*" by \$$x$. In the arithmetic solution the phrase "*the cost of the hat*" stands for a certain unknown number of dollars, which number of dollars is represented by a single letter, x, in the algebraic solution.

2. The sum of two numbers is 72 and one number is twice as large as the other. What are the numbers?

Let	x=the smaller number.
Then	$2x$=the larger number.
Therefore	$2x+x=72$,
i. e.	$3x=72$.
Hence	$x=24$, the smaller number,
and	$2x=48$, the larger number.

3. Divide \$65 between A and B so that B shall receive 4 times as much as A.

4. A rectangle is 3 times as long as it is broad, and the distance around it is 64 feet. Find the length and breadth of the rectangle.

5. A piece of timber 18 feet long must be cut so as to give one piece 2 feet long and two other pieces, one of which must be 3 times the length of the other. Find how long each one of these pieces will be.

6. A father said to his son, "Next year the sum of our ages will be 70 years, and I will be 4 times as old as you will be." What is the present age of each?

7. A man has $48, consisting of an equal number of bank notes of the denominations of $1, $2, and $5. What number has he of each?

Let x=the number he has of each. Then in $1 bills he has $\$x$, in $2 bills he has $\$2x$, and in $5 bills he has $\$5x$.

8. John, Henry, and Mary paid $3.60 for their books. Henry's cost twice as much as John's, and Mary's cost three times as much as John's. Find the cost of each scholar's books.

9. If you add together 3 times and 5 times and 7 times a certain number you will obtain 315. What is the number?

10. Three persons subscribed $50000 to build a hospital. The first two subscribed equal amounts, but the third party subscribed 2 times as much as either of the others. How much did each person subscribe?

11. A man bought 10 turkeys, 10 chickens, and 10 ducks, paying $10 for all. A chicken cost the same as a duck, and a turkey cost 3 times as much as a duck. What was the cost of each?

12. A, B, and C form a partnership to do business. A furnishes 4 times as much capital as C, and B furnishes 3 times as much as C. Altogether the three men put in $24000. What amount is furnished by each?

13. A man and three boys did a piece of work for $16. How should this money be divided among them, if we suppose that the man did twice as much work as each boy?

14. A man has a farm of 240 acres, of which there is twice as much marsh land as wood land, but the rest of the farm is 3 times as large as the marsh land and wood land taken together. Find the number of acres each of marsh land and wood land in the farm.

15. Divide $1100 among A, B, and C so that A may have twice as much as B, and B three times as much as C.

16. A man raised 1730 bushels of grain, of which there was 3 times as much oats as wheat, and 2 times as much corn as oats. Find the amount of each kind that he raised.

17. On a certain day a storekeeper took in $360, of which there was 4 times as much in bank notes as in coin, and 5 times as much in silver as in gold. Find the amount of each kind of money that he received.

18. Of three brothers the middle one is 4 times and the oldest one 5 times as old as the youngest. The sum of the ages of the two oldest is 36 years. Find the age of each of the brothers.

19. If each year I should double the money that I had at the beginning of that year, in five years from now I would have $63000. How much money have I now?

20. There are three numbers, the sum of the last two of which is 63. The second is five times the first, and the third equals the difference between the first and second. What are the numbers?

EXERCISE C.

EXPRESSIONS IN WHICH SEVERAL LETTERS ARE USED TO STAND FOR NUMBERS.

11. We have already learned that any letter may be used to represent any number, but it often happens that

different numbers occur in the *same* question. In this case *different* letters may be used to represent these numbers, but here, as before, any *one* letter must represent the *same* number throughout question and answer.

Find the value of the following expressions, if $a=5$, $b=4$, $c=3$, $d=2$, $e=1$:

1. $9a+2b+3c-2e$.
2. $4e+3a-3b+5c$.
3. $2a+7b-e-c+7$.
4. $5a-b-c-3d-6e$.
5. $10e-2c+9b-a+d$.
6. $20d-6c-2a$.

12. Just as $6\times b$ is written $6b$, so if we wish to express the product of two numbers represented by a and b respectively, we would write it merely ab, instead of $a\times b$. Also abc means just the same as $a\times b\times c$. This custom of omitting the sign $\times$ between a figure and a letter, or between two letters, is universal in Algebra.

Find the value of each of the following six expressions, if $a=2$, $b=4$, $c=6$, $d=\frac{1}{2}$, $e=5$:

7. $7ae+3bc+9e$.
8. $3abd-cd-ab+2$.
9. $2bcd+3ec-a-5bde$.
10. $4abcd+3bcd-cde+5d-3\frac{1}{2}$.
11. $bce+4dab-bade+10-bc$.
12. $abcde+10+bcd+9-dce+8$.

Find the value of each of the following eight expressions, if $a=6$, $b=\frac{2}{3}$, $c=4$, $n=\frac{1}{4}$, $s=1$, $t=\frac{3}{2}$:

13. $10at-2cns+6\frac{1}{2}-at$.
14. $ab+bcnt-4s+20n$.
15. $3ct-7cn+2cn-abt$.
16. $3cs-2bc-ct+12\frac{1}{3}$.
17. $aa+ab+ast+cc$.
18. $ccc+aa+ss-4ac$.
19. $aa-2a+ccc-3c+ssss-4s$.
20. $aacc-500-nccc-6ab$.

21. What is the value of bbb, if $b=3$? if $b=5$? if $b=\frac{1}{2}$? if $b=\frac{1}{3}$? if $b=\frac{2}{3}$?

22. If $a=2$, what is the value of aa? of aaa? of $aaaa$? of $aaaaa$? of $aaaaaa$?

13. When a product consists of the same number repeated any number of times, the product is called a **Power** of that number, and is usually written in a simplified form. Thus:

aa is written a^2, read *a square* or *second power of a*;
aaa is written a^3, read *a cube* or *third power of a*;
aaaa is written a^4, read *a fourth* or *fourth power of a*;
and so on.

The small figure written above and to the right of a number is called the **Exponent** or **Index** of the power; it shows how many times the number occurs in the power.

According to this notation, a^1 would mean that a is used once as a factor, but when a number is used only once as a factor it is customary to omit the exponent altogether and write simply a instead of a^1.

23. Write 3 times x square; 5 times y cube; 5 times the fourth power of b; 7 times the fourth power of b; 9 times the fourth power of b; a times the fourth power of b.

24. How would you abbreviate $bbxxx$?

Just as $a \times b$ is written ab, so $b^2 \times x^3$ is written b^2x^3.

25. Write in the abbreviated form expressions 17 to 20 inclusive.

26. Find the value of $4a^3b^2x^4$, if $a=4$, $b=3$, and $x=1$.

Find the value of each of the following five expressions, where $a=2$, $b=3$, $c=1$, $d=6$:

27. $a^2+b^2+c^2+d^2$.
29. $a^4+b^4-c^4$.
28. $2a^2+3b^3-4c^4$.
30. $3abc-b^2c-6c^3$.
31. $a^3+3a^2b+3ab^2+b^3$.

14. When several numbers are multiplied together the result is called the **Product**, and each of the numbers

or the product of any number of them is called a **Factor** of the product. Thus, if 5, a, b, b are multiplied together the product is $5ab^2$, and the factors of the product are 5, a, b, b^2, $5a$, $5b$, ab, $5ab$, $5b^2$, ab^2.

Any factor of an expression is called the **Coefficient** or **Co-factor** of the product of the remaining factors. Thus, in the expression a^2bc, a^2 is the coefficient or co-factor of bc, a^2b is the coefficient of c, ab is the coefficient of ac, a^2c is the coefficient of b, and a is the coefficient of abc. Similarly, bc, c, ac, b, and abc are the coefficients of a^2, a^2b, ab, a^2c, and a, respectively.

When a product is made up partly of numbers represented by figures and partly of numbers represented by letters, as $2 \times 3ab^2$, it is sometimes convenient to distinguish between these two kinds of factors. In this case we call the numbers represented by figures *numerical* factors and those represented by letters *literal* factors.

Of course the product of all the numerical factors is the coefficient of the product of all the literal factors, and the former is often referred to as the **Numerical Coefficient.**

32. In abx^2 what is the coefficient of x^2? of bx^2? of ab? of abx? of a?

33. In $12abn$ what is the numerical coefficient? of what is it the coefficient? What is the coefficient of bn? of $2bn$? of $4abn$? of 3?

34. Write five different factors of $15cdx^2$, and tell of what each factor named is the coefficient.

35. Write five different factors of $\frac{4}{9}a^3ny^2$, and tell of what each factor named is the coefficient.

15. In Algebra, as in Arithmetic, the symbol for **Divided by** is $\div$.

More often, however, division is represented by means of a fraction where the dividend is written above the line and the divisor below.

Thus, $a \div b$ is written $\frac{a}{b}$, and when written in this form it is often read "*a over b.*"

Write the following in the usual algebraic notation, as explained in Arts. 10, 13, and 14.

36. A known number divided by 6.

37. A known number divided by another known number.

38. An unknown number divided by three times a known number.

39. Four times the square of a known number divided by 5 times the cube of an unknown number.

40. Twice the square of a known number times the cube of an unknown number over the product of two unknown numbers.

Find the value of each of the following expressions, if $a=2$, $b=5$, $m=8$, $s=\frac{1}{2}$, $t=\frac{4}{3}$:

41. $\frac{8ab}{5m}+\frac{40b}{5m}$.

42. $\frac{a^2}{m}+\frac{m}{b}$.

43. $\frac{m}{a^2}+\frac{b}{m}$.

44. $\frac{a^2}{b}+\frac{b}{a^2}$.

45. $\frac{a^2}{b}+\frac{b}{a^2}$.

46. $\frac{a^2m}{b}+\frac{3}{5as}-5$.

47. $\frac{9s^2t}{2}+\frac{b^2s^3m}{5}+\frac{3}{4}$.

48. $\frac{1}{2}a^2m-\frac{1}{3}ab+\frac{tm}{a}$.

49. $\frac{a^2}{b}-\frac{3}{ab}+\frac{3t^2s}{10}$.

50. $\frac{a}{s^2}+\frac{t}{m}+\frac{5}{12}$.

CHAPTER II.

UNION OF TERMS AND REMOVAL OF PARENTHESES.

EXERCISE 7.

UNION OF SIMILAR TERMS.

16. When an expression is broken in parts just before each of the signs + and −, each part thus formed is called a **Term** of the expression. Thus, in the expression $3ab+4c^2-2e-15$, the terms are

$$3ab, \quad +4c^2, \quad -2e, \quad -15.$$

We speak of the terms of an expression in a manner somewhat analagous to the way in which we speak of the syllables of a word. A syllable may not convey an intelligible idea when taken by itself, but when joined with other syllables to make up a word, the whole word does convey a definite idea. So a *term* taken by itself may not, at this stage, express any idea, but the whole expression does convey an idea. See Art. 7. Indeed, each term would, even now, convey a definite idea were it not for the sign + or − which always goes with each term after the first.

17. The terms of an expression which have the sign + or no sign at all are called **Additive Terms**, and those terms which have the sign − are called **Subtractive Terms.** Thus, in the expression $n+2b-4a+5a-7b$, the terms n, $+2b$, and $+5a$ are additive terms, and $-4a$ and $-7b$ are subtractive terms.

In comparing several additive terms they are spoken of as terms of the *same sign*, and several subtractive terms are also spoken of as terms of the same sign; but when

additive and subtractive terms are spoken of together, they are said to be terms of *unlike* or *opposite signs*.

Notice that when we speak of *the signs*, without any further qualification, that it is *only* the *first two* of the four fundamental signs of algebra, $+$, $-$, $\times$, and $\div$, that we have reference to.

18. Terms whose literal parts are identical and whose signs and numerical coefficients may or may not differ are called **Similar Terms.** Thus, in the expression $9a^2b-3ab^2-a^2b+4ab^2$, the terms $9a^2b$ and $-a^2b$ are similar; also the terms $-3ab^2$ and $+4ab^2$ are similar; but $9a^2b$ and $+4ab^2$ are *not* similar.

1. Is the expression $12+2-8+5-4$
the same as $12+2+5-8-4$?
Is the last equal to $19-8-4$?
Is this the same as $19-12$?

2. Is the expression $2+4-5+20-14+11$
the same as $2+4+20+11-5-14$?
Is the last equal to $37-5-14$?
Is this the same as $37-19$?

3. Is the expression $8a-2a-4a+5a-2a$
the same as $8a+5a-2a-4a-2a$?
Is the last equal to $13a-2a-4a-2a$?
Is this the same as $13a-8a$?

4. Is the expression $n+5ab^2-2ab^2+15ab^2-7-12ab^2$
the same as $n+5ab^2+15ab^2-2ab^2-12ab^2-7$?
Is the last equal to $n+20ab^2-2ab^2-12ab^2-7$?
Is this the same as $n+20ab^2-14ab^2-7$?

5. In *any expression whatever*, will the value of the expression be changed, if all the additive terms are written first and the subtractive terms following?

6. In any expression, can any number of similar additive terms be replaced by a single additive term?

7. In any expression, can any number of similar subtractive terms be replaced by a single subtractive term?

8. A man had n dollars. He gained 23 dollars and then lost 9 dollars. How much did he then have?

How much more is $n+23-9$ than n?

Write an expression of two terms which shall be equal to $n+23-9$.

The two terms $+23-9$ can be replaced by what single term?

9. A man had n dollars. He gained 9 dollars and then lost 23 dollars. How much did he then have?

How much less is $n+9-23$ than n?

Write an expression of two terms which shall be equal to $n+9-23$.

The two terms $+9-23$ can be replaced by what single term?

10. How much more is $a+52-21$ than a?

Write an expression of two terms which shall be equal to $a+52-21$.

The two terms $+52-21$ can be replaced by what single term?

11. How much less is $a+21-52$ than a?

Write an expression of two terms which shall be equal to $a+21-52$.

The two terms $+21-52$ can be replaced by what single term?

12. How much more is $100+17a-5a$ than 100?

Write an expression of two terms which shall be equal to $100+17a-5a$.

The two terms $+17a-5a$ can be replaced by what single term?

13. How much less is $100+5a-17a$ than 100?

Write an expression of two terms which shall be equal to $100+5a-17a$.

The two terms $+5a-17a$ can be replaced by what single term?

14. How much more is $n+29a-11a$ than n?

Write an expression of two terms which shall be equal to $n+29a-11a$.

The two terms $+29a-11a$ can be replaced by what single term?

15. How much less is $n+11a-29a$ than n?

Write an expression of two terms which shall be equal to $n+11a-29a$.

The two terms $+11a-29a$ can be replaced by what single term?

16. In any expression whatever, can two similar terms, one additive and one subtractive, be replaced by a single term?

What kind of a term is this, if the numerical coefficient of the additive term is the greater?

What kind of a term, if the numerical coefficient of the subtractive term is the greater?

19. The above questions suggest the following principles:

I. The order of terms in an expression is not fixed, the value of an expression being unchanged if all the additive terms be written first, followed by the subtractive terms.

II. Any number of similar additive terms in an expression may be replaced by a single additive term, and any number of similar subtractive terms may be replaced by a single subtractive term.

III. Two similar terms, one additive and one subtractive, may be replaced by a single term in which the numerical coefficient is the difference of the numerical coefficients of the two terms, and the sign is + or − the same as that one of the displaced terms having the greater numerical coefficient.

EXERCISE 8.

EXAMPLES.

In each of the following expressions combine the similar additive terms into a single additive term, and the similar subtractive terms into a single subtractive term. Afterwards combine these two terms into a single term.

1. $7a-3a-4a+6a$.
2. $11x-5x+8x-6x$.
3. $50-6b+2b-20b+8b$.
4. $100+3ab-2ab+5ab-3ab$.
5. $n+12y-18y+2y-6y$.
6. $8g-3g+7g-9g-4g+8g$.
7. $a+20xy-30xy-50xy+10xy$.
8. $a+9a^2-2a^2+5a^2+8a^2-7a^2-9a^2$.
9. $37-10m+13m-15m-8m$.
10. $5m-4m+8m-12m-7m+10m$.
11. $10x^3-7x^3+3x^3+8x^3-12x^3-x^3$.
12. $125+8y^2-3y^2+6y^2-20y^2-4y^2$.
13. $10+5a^2b^2-a^2b^2-7a^2b^2+5a^2b^2-a^2b^2$.
14. $n+20x^2y-5x^2y+x^2y-11x^2y-10x^2y$.
15. $100+13yx^2-23yx^2+33yx^2-43yx^2$.
16. $6a-18b^2c+4b^2c-b^2c+10b^2c$.
17. $50abcd-11abcd-41abcd+3abcd$.
18. $\frac{1}{2}x-\frac{1}{3}x+x+\frac{2}{3}x$.

$$\begin{aligned}\tfrac{1}{2}x-\tfrac{1}{3}x+x+\tfrac{2}{3}x&=\tfrac{1}{2}x+\tfrac{2}{3}x+x-\tfrac{1}{3}x,\\&=\tfrac{3}{6}x+\tfrac{4}{6}x+\tfrac{6}{6}x-\tfrac{2}{6}x,\\&=\tfrac{11}{6}x-\tfrac{2}{6}x,\qquad\text{etc.}\end{aligned}$$

19. $6+\frac{3}{2}a+\frac{3}{5}a-\frac{1}{2}a$.
20. $1-\frac{1}{10}a+\frac{3}{5}a+\frac{1}{2}a-a$.
21. $5b+\frac{1}{4}b-\frac{3}{2}b+2b-\frac{1}{2}b+\frac{7}{4}b$.
22. $10a-3b+\frac{1}{2}b+\frac{3}{4}b-b-\frac{1}{4}b$.

Shorten the following expressions as much as possible by a careful grouping and uniting of similar terms:

23. $99997+83752+3$.

24. $9999+9998+9996+9995+4+2+1+5$.

25. $17+18+19+20+21+22+23$.

26. $3a+7b+5a$.

27. $x+5y+8x$.

28. $24p+36q+17q+35p$.

29. $14n+27ab+8n+13ab+16n$.

30. $3a+5b^2+3dc+6dc+17a+11b^2+9b^2+2a+dc$.

31. $5a+3b-3b$.

32. $9b+3a-9b$.

33. $4a-x+x$.

34. $84589+8783-4589$.

35. $28654+9999-18654$.

36. $7a+5b-4a$.

37. $9a-5b+a$.

38. $20-7x+9$.

39. $57-7a-9$.

40. $30+6c+5$.

41. $18a-16bc+10a+14bc-16a-2bc$.

42. $10a^2-8ab-3ab-6a^2+12ab$.

43. $2xy-6z+10xy-8z+16z-12xy$.

44. $8m-10ts+6n-4ts-2n+2m-8ts-6m$.

45. $5a+8b^2c-7d-2a-9b^2c+2d-2a+2b^2c+6d$.

46. $10m+11-5x-12-4m-3x+1+9x-5m$.

47. $9a-7b+3c-8a+7b-3c-5b-8c$.

48. $9a^3b^3c-7a^3b^2c^2+10a^3b^2c^2-5a^3b^3c$.

49. $11xy+2ab-4xy-25ab+ab+10$.

50. $25-25x+25y+13-30y+20x-8$.

51. $n^2-2+r^2+2-16r^2+9n^2+5+n^2$.

52. $13x-5y+8z-5x+9y-11z-3x-6y+z$.

53. $54m-62n+18x-62m-6x+42n+10m+18n-14x$

54. $10m+11-5a^2-12+4m-6a^2+1+18a^2-10m$.

55. $a-2bc+18cd-14bd-21cd-3bc+5cd$.

56. $9a^4x^2-54a^5x+27a^6-7a^4x^2+13a^5x-a^5x+3a^6$.

EXERCISE 9.

PARENTHESES AND THEIR REMOVAL.

20. If we wish to indicate that the sum of the numbers 2 and 3 is to be subtracted from 12, we write it like this,

$$12-(2+3),$$

using a *parenthesis* to enclose the expression $2+3$.

The use of the parenthesis in Algebra is quite frequent; and it always serves to show that the expression which it embraces, whether simple or complicated, is to be looked upon as a *single number* just as though it were represented by a single symbol.

Thus, $(3+2)\times(7-5)$, or $(3+2)(7-5)$, means the same as 5×2; $(3+2)^2$ means the same as 5^2, that is, 25; also $(3a)^2$ means *the square of* 3 *times* a, but $3a^2$ means 3 *times the square of* a. The student will readily notice the difference between $(3a)^2$ and $3a^2$.

Expressions containing parentheses should be read in a manner so as to show what numbers are enclosed in parentheses. Thus, $(3+2)(7-5)$ may be read, "parenthesis 3+2 multiplied by parenthesis 7−5," but should not be read, "3 plus 2 times 7 minus 5," for this last would be the way to read the expression $3+2\times 7-5$.

1. Write 16 diminished by the sum of the numbers 2, 3, and 4.

2. Write a diminished by the sum of the numbers b, c, and d.

3. Write 3 times a minus the sum of the numbers 2 times b, 3 times c, and 4 times d.

4. Write a plus the square of the sum of the numbers b and c.

Find the value of each of the following expressions, 5 to 11, where $m=6$, $t=5$, $p=4$, $q=3$, and $r=2$:

5. $(m-t)\times(p+q)-r$.
6. $2(m+t+p)+(q-r)$.
7. $m+(t-p)-(q+r)$.
8. $(3m-t)-(p+q+r)$.
9. $m+2(t-p)-(q-r)$.
10. $(m^2-t^2)-(p+2q)+r^2$.
11. $2(m^2-t^2)+\frac{1}{2}(p+2q)+r^2$.

Find the value of the following three expressions, where $a=1$, $b=2$, $c=3$:

12. $(2a+b)^2+4(2c-2b)^3$.
13. $3(a+b)^2-(b+c)^2$.
14. $6(a+b)c-(b+c)abc$.

EXERCISE 10.

REMOVAL OF PARENTHESES.

GENERAL FORM $a+(b+c)$.

1. How much greater is $10+(7+3)$ than $10+7$?
2. How much greater is $12+(7+3)$ than $12+7$?
3. How much greater is $25+(7+3)$ than $25+7$?
4. How much greater is $a+(7+3)$ than $a+7$?
5. How much greater is $a+(8+3)$ than $a+8$?
6. How much greater is $a+(b+3)$ than $a+b$?
7. How much greater is $a+(b+5)$ than $a+b$?
8. How much greater is $a+(b+c)$ than $a+b$?
9. How much must be added to $a+b$ to equal $a+(b+c)$?
10. Write an expression equal to $a+(b+c)$ without using a parenthesis.

GENERAL FORM $a+(b-c)$.

11. How much less is $10+(7-3)$ than $10+7$?
12. How much less is $12+(7-3)$ than $12+7$?
13. How much less is $25+(7-3)$ than $25+7$?
14. How much less is $a+(7-3)$ than $a+7$?
15. How much less is $a+(8-3)$ than $a+8$?

16. How much less is $a+(b-3)$ than $a+b$? Why?

17. How much less is $a+(b-5)$ than $a+b$? Why?

18. How much less is $a+(b-c)$ than $a+b$?

19. How much must be subtracted from $a+b$ to equal $a+(b-c)$?

20. Write an expression equal to $a+(b-c)$ without using a parenthesis.

GENERAL FORM $a-(b+c)$.

21. How much less is $10-(7+3)$ than $10-7$?

22. How much less is $12-(7+3)$ than $12-7$?

23. How much less is $25-(7+3)$ than $25-7$?

24. How much less is $a-(7+3)$ than $a-7$? Why?

25. How much less is $a-(8+3)$ than $a-8$? Why?

26. How much less is $a-(b+3)$ than $a-b$? Why?

27. How much must be subtracted from $a-b$ to equal $a-(b+c)$?

28. How much less is $a-(b+5)$ than $a-b$?

29. How much must be subtracted from $a-b$ to equal $a-(b+5)$?

30. Write an expression equal to $a-(b+5)$ without using a parenthesis.

31. How much less is $a-(b+c)$ than $a-b$? Why?

32. How much must be subtracted from $a-b$ to equal $a-(b+c)$?

33. Write an expression equal to $a-(b+c)$ without using a parenthesis.

34. The additive terms b and $+c$ *within* the parenthesis occur as what kind of terms when the expression is written without a parenthesis?

GENERAL FORM $a-(b-c)$.

35. How much greater is $10-(7-3)$ than $10-7$?

36. How much greater is $12-(7-3)$ than $12-7$?

37. How much greater is $25-(7-3)$ than $25-7$?

38. How much greater is $a-(7-3)$ than $a-7$?

39. How much greater is $a-(8-3)$ than $a-8$?

40. How much greater is $a-(b-3)$ than $a-b$?

41. How much must be added to $a-b$ to equal $a-(b-3)$?

42. Write an expression equal to $a-(b-3)$ without using a parenthesis.

43. How much greater is $a-(b-5)$ than $a-b$? Why?

44. How much must be added to $a-b$ to equal $a-(b-5)$?

45. Write an expression equal to $a-(b-5)$ without using a parenthesis.

46. How much greater is $a-(b-c)$ than $a-b$? Why?

47. How much must be added to $a-b$ to equal $a-(b-c)$?

48. Write an expression equal to $a-(b-c)$ without using a parenthesis.

49. The additive term b and the subtractive term $-c$ *within* the parenthesis occur as what kind of terms when the expression is written without a parenthesis?

REMOVAL OF TERMS FROM A PARENTHESIS PRECEDED BY THE PLUS SIGN.

50. When the sum of several numbers is to be found, increasing one of the numbers has what effect on the sum? Decreasing one of the numbers has what effect on the sum?

51. How much less is $a+(b+c-d-f)$ than $a+(b+c-d+e-f)$?

52. How much must be added to $a+(b+c-d-f)$ to equal $a+(b+c-d+e-f)$?

53. Write an expression equal to $a+(b+c-d+e-f)$, where e appears outside the parenthesis.

54. Can c be thus written outside the parenthesis instead of e?

55. Can *any* additive term be thus taken outside of a parenthesis?

56. How much greater is $a+(b+c-d+e)$ than $a+(b+c-d+e-f)$?

57. How much must be subtracted from $a+(b+c-d+e)$ to equal $a+(b+c-d+e-f)$?

58. Write an expression equal to $a+(b+c-d+e-f)$, where $-f$ appears outside the parenthesis.

59. Can $-d$ be thus written outside the parenthesis instead of $-f$?

60. Can *any* subtractive term be thus written outside of a parenthesis?

61. Can *any* term, additive or subtractive, be taken out of a parenthesis?

62. Can any *two* terms be taken out of a parenthesis one after another?

63. Can any number of terms be taken out of a parenthesis?

64. Can all the terms be taken out of a parenthesis?

21. These questions, if rightly answered and understood, lead to the fact, that *when a parenthesis is preceded by a + sign the parenthesis may be erased* and the value will not be changed.

REMOVAL OF TERMS FROM A PARENTHESIS PRECEDED BY THE MINUS SIGN.

22. When the difference of two numbers or expressions is to be found, the number or expression from which something is subtracted is called the **Minuend** and the number or expression subtracted is called the **Subtrahend** and the result is called the difference, or **Remainder.**

65. When the difference of two numbers is to be found, increasing the subtrahend has what effect on the difference? Why?

Decreasing the subtrahend has what effect on the difference? Why?

66. How much greater is $a-(b+c-d-f)$ than $a-(b+c-d+e-f)$?

67. How much must be subtracted from $a-(b+c-d-f)$ to equal $a-(b+c-d+e-f)$?

68. Write an expression equal to $a-(b+c-d+e-f)$, where $+e$ does not appear in the parenthesis.

69. When the additive term $+e$ is taken out of the parenthesis, it becomes what kind of a term?

70. Can $+c$ be removed from the parenthesis instead of $+e$? If it is so removed, what kind of a term does it become?

71. Can *any* additive term be thus removed from a parenthesis preceded by a minus sign?

What kind of a term does it become when it is thus removed?

72. How much less is $a-(b+c-d+e)$ than $a-(b+c-d+e-f)$?

73. How much must be added to $a-(b+c-d+e)$ to equal $a-(b+c-d+e-f)$?

74. Write an expression equal to $a-(b+c-d+e-f)$, where the term $-f$ does not appear in the parenthesis.

75. When the subtractive term $-f$ is removed from the parenthesis, it becomes what kind of a term?

76. Can $-d$ be thus removed from the parenthesis instead of $-f$? When thus removed, it becomes what kind of a term?

77. Can *any* subtractive term be thus removed from a parenthesis preceded by a — sign?

78. Can *any* term, additive or subtractive, be removed from a parenthesis preceded by a — sign?

79. Can any *two* terms be removed, one after another, from a parenthesis preceded by a — sign?

80. Can any number of terms be removed from a parenthesis preceded by a — sign?

81. Can all the terms be removed from a parenthesis preceded by a — sign?

82. When all the terms are removed from a parenthesis preceded by a — sign, the additive terms become what kind of terms? The subtractive terms become what kind of terms?

23. These questions, if rightly answered and understood, lead to the fact that whenever any parenthesis is preceded by a — sign, all the terms within the parenthesis may be taken out, provided that in doing so each additive term be changed to a subtractive term and each subtractive term be changed to an additive term; or, what comes to the same thing, *whenever a parenthesis is preceded by a — sign, the parenthesis may be erased, provided all the terms within the parenthesis be changed from additive to subtractive or from subtractive to additive as the case may be.*

Changing all the additive terms of an expression into subtractive, and all the subtractive into additive, is spoken of in algebra as changing all the *signs* of the expression, although it will be observed that the first term of an expression has no sign at all. While this latter way of speaking is not quite so definite as the former, still it is much shorter and is very convenient.

24. We have already considered the case of parentheses, preceded by the sign + and also by the sign −, but when a parenthesis stands at the beginning of an expression, it is usually not preceded by any sign at all. This case, however, is so simple that it is very easily settled.

In an expression like $(a+b-c)-d$	In an expression like $a+b-c-d$
we are first to add together the numbers a and b, and from this sum subtract c, and then from this difference subtract d.	we are first to add together the numbers a and b, and from this sum subtract c, and then from this difference subtract d.

From these two statements, which are exactly alike, it is evident that when a parenthesis stands at the beginning of an expression, the expression means exactly the same thing that it would if the parenthesis were erased, and from this it is evident that *when a parenthesis is preceded by no sign at all, the parenthesis may be erased*, just the same as though the parenthesis were preceded by the sign +, or in other words, when a parenthesis is preceded by no sign at all, it is treated just as though it were preceded by the sign +.

EXERCISE 11.

MISCELLANEOUS EXAMPLES ON THE REMOVAL OF PARENTHESES.

Write each of the following expressions without using any parentheses :

1. $m+n+(p+q+r)$.
2. $m+n+p-(q+r)$.
3. $(a-b)+(p-q)-10$.
4. $(a+b+c)-(q-r)$.
5. $a+(a^2-b)-(b^2+c)$.
6. $a-b-(a^2+b^2-c^2)$.
7. $(x+2y)-(a+2b-2c)$.
8. $(3ab-a)+(3ab^2-2cd)-(cd^2-a^2)$.
9. $(4m-2n)-(3p-2q+r)$.
10. $m-(a+b)-(c-d)$.
11. $(m^2+n)-(p^2-q)-(a^2-10)$.
12. $(m^2-n^2)-(p+2q)+(6r^2+ab-ab)$.
13. $(a^2+b^2+ab)-(av+5bc-abc)$.
14. $(7ax-3by)-(5ax-by)-10$.
15. $6a^2x-(13by^2-5abxy-9y^4)+(7x^3-8y^3)$.
16. $21-(5-8y+13y)-(6+16y-15x)$.
17. $75-(15-8y+21y)-(30+80y-15x^2)$.
18. $(3ax-5y)-(2ay-\frac{5}{2}x)+(7z-3y)$.

CHAPTER III.

ADDITION.

EXERCISE 12.

ADDITION OF EXPRESSIONS.

25. When two or more expressions are to be added together, each expression may be enclosed in a parenthesis, these parentheses written one after another, separated by plus signs. Thus, if

$$a+2, \quad 3a-1 \text{ and } a+3$$

are to be added, we would indicate the sum by

$$(a+2)+(3a-1)+(a+3).$$

These parentheses may now be removed, and the sum indicated thus: $a+3a+a+2-1+3$

which, by uniting the terms, may be written

$$5a+4.$$

Let us find the sum of the three expressions

$$x+y, \quad x-z \text{ and } 2x+3y-2.$$

First, we enclose these in parentheses, write them one after another separated by plus signs, and get

$$(x+y)+(x-z)+(2x+3y-2).$$

Second, we remove parentheses as in exercise 10, and get

$$x+y+x-z+2x+3y-2.$$

Third, we arrange these terms so that similar terms shall come together, and get

$$x+x+2x+y+3y-z-2.$$

Fourth, we unite each group of similar terms into a single term and get $4x+4y-z-2,$

and this is the simplest form possible for the sum of the three given expressions.

26. It is easy to see that we could find the sum of any number of expressions, whatever those expressions may be, in a manner similar to that just pursued, viz.: enclose each expression in a parenthesis, write these parentheses one after another separated by plus signs, and then remove parentheses from the expression. Next arrange the terms of the expression thus found, so that similar terms shall come together, and then by uniting each group of similar terms into a single term, we obtain the simplest form possible for the required sum of the given expressions.

EXAMPLES.

Find the sum of the following expressions:

1. $x-3y-4$, $2x-5y-6$ and $x+3y+12$.
2. $3a^2-2ab+5b^2$, $a^2-4ab-3b^2$ and a^2+b^2.
3. $4xy-y^2+23$, $6xy+2y^2-50$ and $3y^2-xy-2$.
4. $x^2+2xy+y^2$, $x^2-2xy+y^2$ and x^2-y^2.
5. $6m-4st-r^2$, $2st+2m+6r^2$ and $10st-7r^2-m$.
6. $3x^2+xy+3y^2$, $x^2-3xy+y^2$ and $3x^2+3y^2$.
7. $a+b$, $12c+3d$, $6b-4d$ and $10a-9b$.
8. $a+4b$, $b+4c$, $c+4d$ and $d+4e$.
9. $3xy+2y$, $5xy-x$, $3x-5y$ and $7xy-x-2y$.
10. $3a-4b-6cd+2e$, $10b+3e-10cd$ and $9a-20e+14cd$.
11. $6c+7-4a-5b$, $6b+7c-4-5a$, and $6a+7b-4c-5$.
12. a^3+b^3, a^3-b^3 and $ab^2+a^2b+b^3$.

EXERCISE 13.

ARRANGEMENT OF WORK IN ADDITION.

27. In finding the sum of two simple expressions by the method already learned, we place these expressions in parentheses, writing them one after another, separating

them by plus signs. But evidently it comes to the same thing if, instead of writing the expressions one *after* another *within* parentheses, we write them one *below* another *without* parentheses, arranging the similar terms in the same vertical column. We may then draw a line under the last expression, and the example is arranged in exactly the same form as in Arithmetic. Now the similar terms in each column may be combined into a single term, as we already know, and this term placed under the line as one term of the sum. When all the columns are thus treated all the terms of the sum are found. Thus, suppose we are required to find the sum of

$$2a^2+3b^2-5cd,\quad 6a^2-2b^2 \text{ and } 8b^2-4a^2-4cd.$$

By the former method we write

$$(2a^2+3b^2-5cd)+(6a^2-2b^2)+(8b^2-4a^2-4cd).$$

Removing parentheses we get

$$2a^2+3b^2-5cd+6a^2-2b^2+8b^2-4a^2-4cd.$$

Arranging similar terms one after another we get

$$2a^2+6a^2-4a^2+3b^2-2b^2+8b^2-5cd-4cd.$$

Uniting similar terms, we get

$$4a^2+9b^2-9cd.$$

By the latter method we write

$$\begin{array}{r} 2a^2+3b^2-5cd \\ 6a^2-2b^2 \\ -4a^2+8b^2-4cd \\ \hline 4a^2+9b^2-9cd \end{array}$$

Now it is very evident that we have here exactly the same terms to combine that we had by the other method, after the parentheses had been removed, and the similar terms brought together. Moreover, it is apparent that these terms are combined in exactly the same order, giving

of course the same result as before; and the only difference between the two methods consists in the arrangement of the work. The similar terms being rather easier to combine in the second arrangement, it is the more convenient arrangement for the beginner.

EXERCISE 14.

EXAMPLES.

1.

$$\begin{array}{r} a^2+ ab+ b^2 \\ 3a^2-3ab-7b^2 \\ \hline 4a^2+5ab+9b^2 \end{array}$$

2.

$$\begin{array}{r} 5ab+6bc-7ca \\ 8ab-4bc+3ca \\ \hline 2ab- bc+5ca \end{array}$$

3.

$$\begin{array}{r} 2x^2-2xy+3y^2 \\ 3x^2+5xy+4y^2 \\ \hline x^2-2xy-9y^2 \end{array}$$

4.

$$\begin{array}{r} 5ax-7by+ cy \\ 3ax+9by-3cy \\ \hline ax-3by- cy \end{array}$$

5. Find the sum of $5h+4t+3u$, $2h+2t+u$, and $7h+3t+4u$.

If h be supposed equal to 100, t equal to 10 and u equal to unity, this example illustrates nicely the analogy between the work the student is now doing and the ordinary addition of numbers in Arithmetic. Thus,

	Hundreds.	Tens.	Units.
$5h+4t+3u$	5	4	3
$2h+2t+u$	2	2	1
$7h+3t+4u$	7	3	4
$14h+9t+8u$	14	9	8

6. Arrange and add $x+y+z$, $2x+3y-2z$, and $3x-4y+z$.

7. Add $14a-6b+3c-5d$ and $9a+7b-4c-9d$.

8. Add $a+b-c+d$ and $a-b-c+3d$.

9. Add $5a^2-3ab+7b^2$, $7a^2+4ab-5b^2$, and $a^2-9ab+4b^2$.

10. Add $a+2b+3c$, $2a-b-2c$, $b-a-c$, and $c-a-b$.

Arrange thus:

$$\begin{array}{r} a+2b+3c \\ 2a-\ b-2c \\ -\ a+\ b-\ c \\ -\ a-\ b+\ c \\ \hline \end{array}$$

11. Add $6b+8c-5a$, $8a-3b+4c$, and $7b-15c-2a$.

12. Add $3n+2r+3s-4t$, $3r-4s-5t-2n$, and $5s-6t+12n-10r$.

13. Add $x+3a+2b-c$, $2y-3b+2c+a$, and $3z+3c-2a-b$.

$$\begin{array}{rrrr} 3a+2b-\ c+x & & \\ a-3b+2c\ \ +2y & & \\ -2a-\ b+3c & & +3z \\ \hline \end{array}$$

14. Add $3x^3-4x^2-x+7$, $2x^3+x^2+3x-10$, $2x^2-7x^3-2x-14$, and $3x^3-12x^2+12+5x$.

In arranging expressions which involve different powers of the same number, it is usual to place all of the terms containing the highest power of the letter in the first column, all the terms containing the next highest power in the next column, and so on. Thus, this example would generally be arranged in this way:

$$\begin{array}{r} 3x^3-\ 4x^2-\ x+\ 7 \\ 2x^3+\ \ x^2+3x-10 \\ -7x^3+\ 2x^2-2x-14 \\ 3x^3-12x^2+5x+12 \\ \hline \end{array}$$

15. Add x^3+4x^2+5x-3, $2x^3-7x^2-14x+5$, and $x^2-x^3-2+10x$.

16. Add $3x^2-4x^3+x^4$, x^3+x^2+x, $4x^4+5x^3$, and $2x^2-3x-4x^3$.

17. Add $5x^3-3x^2+3x-8$, x^2-3x^3+31, $2x^3-8x-5x^2+2$, and $4x+7x^2-9-3x^3$.

18. Add $3x^2-4xy+y^2+2x+3y$, $10xy+8y^2+9y$, and $5x^2-6xy+3y^2+7x-7y$.

19. Add $4ab^2+4cde-6h^3$, $2ab^2+3h^3$, $9cde-7ab^2$, and $ab^2+2cde-5h^3$.

20. Add $4xy+5de-2fg$, $3fg-2xy$, $8xy-3de$, $8de-5xy-2fg$, and $6fg-2xy$.

21. Add $14\frac{5}{6}a^3-7a^2b+6\frac{4}{5}ab^2+11\frac{1}{3}b^3$ and $7\frac{1}{2}a^3-2\frac{1}{3}a^2b-4ab^2-12b^3$.

22. Add $9\frac{1}{2}c+3\frac{3}{5}d+7.75$ and $7\frac{3}{4}c-14\frac{4}{10}d-8$.

23. Add $\frac{5}{6}a+b$, $\frac{1}{3}b-\frac{1}{2}a$, $\frac{1}{3}a-3b$, and $3a-9\frac{1}{9}b$.

24. Add $a^3+3a^2b+3ab^2+b^3$, $a^3-3a^2b+3ab^2-b^3$, $b^3-3ab^2+3a^2b-a^3$, and $a^3+6a^2b+6ab^2+b^3$.

25. Add $12y-5a-7ax$, $5ax+a-8y$, $9a-y-ax$, $4ax-3a+5y$, and $y+a-ax$.

26. Add $a^3+5ab^2+b^3$, $a^3-10ab^2+b^3$, and $5ab^2-2b^3-2a^2b$.

27. Add $x^3-2ax^2+a^2x+a^3$, x^3+3ax^2, and $2a^3-ax^2-x^3$.

28. Show that if $x=a+2b-3c$, $y=b+2c-3a$, and $z=c+2a-3b$, then will $x+y+z=0$.

29. Add $3a+2b-c$, $3b+2c-a$, and $3c+2a-b$.

30. Add $2a-3b^2+c^2$, b^2-2c^3+3d, and $3c^3-2d-a$.

CHAPTER IV.

SUBTRACTION.

EXERCISE 15.

SUBTRACTION OF EXPRESSIONS.

28. When one expression is to be subtracted from another, we may enclose each expression in a parenthesis and separate the minuend from the subtrahend by a minus sign. Thus, if $a+2$ is to be subtracted from $3a-1$, we would indicate the difference by

$$(3a-1)-(a+2).$$

These parentheses may now be removed, care being taken that all the signs in the second parenthesis be changed when the parenthesis is removed, and the remainder written thus, $3a-1-a-2$.

Now by grouping the terms, the same result may be written $3a-a-1-2$,

which by uniting the terms may be written

$$2a-3.$$

which is the required difference.

Let us find the difference between

$$3x^2-4xy+5y^2 \text{ and } 2x^2-2xy-4y^2.$$

First, we enclose each of these expressions in a parenthesis, writing the subtrahend after the minuend, with a minus sign between them, and get

$$(3x^2-4xy+5y^2)-(2x^2-2xy-4y^2).$$

Second, we remove each of these parentheses, taking care to change all the signs within the second parenthesis, and get

$$3x^2-4xy+5y^2-2x^2+2xy+4y^2.$$

Third, we arrange these terms so that similar terms shall come together, and get

$$3x^2-2x^2-4xy+2xy+5y^2+4y^2.$$

Fourth, we unite each group of similar terms into a single term, and get

$$x^2-2xy+9y^2,$$

and this is the simplest form possible for the difference of the two given expressions.

It is easy to see that we could find the difference of any two expressions, whatever these expressions may be, in a manner similar to that just pursued, namely, enclose each expression in a parenthesis, write the minuend first and the subtrahend second, with a minus sign between them, and then remove parentheses. Next arrange the terms of the expression thus found so that similar terms shall come together, and, finally, by uniting each group of similar terms into a single term we obtain the difference of the two given expressions in the simplest form possible.

EXAMPLES.

1. From $17a+4b-3c$ take $3a+3b-c$.
2. From $3x-9y+20z$ take $x-4y+3z$.
3. What is the difference between $6d+25c$ and $6d-25c$?
4. What is the difference between $8x^2+4y$ and $8x^2-4y$?
5. From $x^2+2xy+y^2$ take $x^2-2xy+y^2$.
6. From $4x^2+2xy+3y^2$ take $x^2-xy+2y^2$.
7. From $6x^2-13x-7$ take $3x^2+5x-2$.
8. From $2x-11a+10b-5c-23$ take $5a+2c-10-3b$.
9. From $2x^3+x^2-35x+49$ take $x^3-29x+42$.
10. From $4x^4-3x^3-2x^2+7x+9$ take $x^4-2x^3-2x^2-7x-9$.
11. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$
12. From $3x+10$ take $10-3y$.

EXERCISE 16.

ARRANGEMENT OF WORK IN SUBTRACTION.

29. In finding the difference of two expressions by the method already learned, we place each expression in a parenthesis, writing the subtrahend *after* the minuend with a minus sign between them, and then remove the parentheses. But, evidently, it comes to the same thing if, instead of writing the subtrahend with all its signs changed *after* the minuend, we write the subtrahend with all its signs changed *below* the minuend, placing similar terms of the minuend and subtrahend in the same veritical column, and then uniting similar terms exactly as in addition. For example, if we wish to subtract

$$2a^2-4b^2-6 \text{ from } 9a^2+3b^2-7$$

we arrange the work thus:

Minuend	$9a^2+3b^2-7$
Subtrahend with signs changed	$-2a^2+4b^2+6$
Remainder	$7a^2+7b^2-1$

The signs of the subtrahend need not actually be changed if the student will *imagine* them changed as he proceeds in the work.

EXERCISE 17.

EXAMPLES.

	1.	2.
From	$6x-14y+10$	$9a-4b+8c$
take	$2x-6y+6$	$a+2b+3c$

	3.	4.
From	$9a-8b+7c-3d$	$7x-2y+3z-4u$
take	$5a-6b-3c+2d$	$5x+3y-5z+8u$

5.

From $4x-3y+9u-8v$
take $2x+4y-3u-8v$

6.

$m-3n+p-7$
$m-4n-p+8$

7.

From $3\frac{1}{2}a-2\frac{1}{3}b+3\frac{1}{2}c$
take $1\frac{1}{2}a+1\frac{1}{3}b+4\frac{1}{2}c$

8.

$a-3b+4c-2d$
$7b-2c-8d+e$

9.

From $15a-7b+3c-7d-8e$
take $10a+7b-3c+4d+4e$

10.

$7x-2y-z+4+a$
$x+y+5z-2+n$

11.

From $73a-52b-71c+21d-52x+17y+59z+11$
take $54a-60b+81c+37d+18x-33y+99z+7$

12. From $4x^2+2xy+3y^2$ take $x^2-xy+2y^2$.

13. From $a^3+3a^2b+3ab^2+b^3$ take $a^3-3a^2b+3ab^2-b^3$

14. From $2x+11a+10l-5c-23$ take $2c-10+5a-3b$.

15. From $3x^3-2x^2+3x-4$ take x^3-4x^2-3x+1.

16. From $72x^4-78x^3-10x^2+17$ take $2x^4+36x^3-17x+10x^2$.

17. From $3x^4+5x^3-6x^2-7x+5$ take $2x^4-2x^3+5x^2-6x-7$.

18. From $7x^2-8x-1$ take $5x^2-6x+3+x^2$.

19. From $4x^4-3x^3-2x^2-7x+9$ take $x^4-2x^3-2x^2+7x-9$.

20. From $5x^2+6xy-12xz-4y^2$ take $2x^2-7xy+4xy-3y^2$.

21. From $x^2+3xy-y^2+yz-2y^2$ take $x^2+2xy+5xz-3y^2-2z^2$.

22. From $7x^4-2x^2+2x+2$ take $4x^3-2x^2-2x-14$.

23. From $4x^3-2x^2-2x-14b$ take $2x^3-8x^2+4x+ab$.

24. From $3\frac{1}{2}a-2\frac{1}{3}b+3\frac{1}{2}c-5d$ take $1\frac{1}{2}a+\frac{1}{2}b-4\frac{1}{4}c+3d$.

25. From $6ay-5xy+2a^2x^2$ take $4xy-3ay-a^2x^2$.

26. From $\frac{7}{2}a+\frac{7}{2}b-\frac{9}{4}c-11d+\frac{1}{4}$ take $\frac{5}{3}a-\frac{9}{4}b-\frac{11}{2}c+\frac{1}{2}d-\frac{3}{4}$

27. From $a+2$ take $b-3$.

28. From $a+b$ take $a+c$.

29. From $a+b$ take $c+d$.

30. From $a-b$ take $b-c$.

31. From a^2-ab+b^2 take x^2+xy+y^2.

32. From $4aby^2-5axy+2a^2x^2$ take $a^2x^2-abx-3aby^2$.

33. From $6x^2+7xy-5y^2-12xyz-8yz$ take $8xy-7yz+3x^2-4y^2+5xyz$.

34. From $2x+11a+10b-5c-23$ take $2c-10+5a-3b$.

35. From $4rs+62n^2-26m-23n^2$ take $9n^2-2rs+21m+2n^2-3rs$.

EXERCISE 18.

Additon and Subtraction of Equals.

1. How much less is $x+4$ than $x+5$? If $x+5=12$, what does $x+4$ equal? Why? What does $x+3$ equal? Why? What does $x+2$ equal? What does $x+1$ equal? What does x equal?

2. If $x+5=28$, what does $x+3$ equal? Why? What does $x+1$ equal? Why? What does x equal? Why?

3. If $x+30=50$, what does $x+10$ equal? Why? What does $x+7$ equal? Why? What does $x+3$ equal? Why? What does x equal? Why?

4. If $x+12=44$, write what x equals. What must you do to *each member* of the equation $x+12=44$ to get the equation $x=32$?

5. If $x+23=48$, write what x equals. What must you do to each member of the equation $x+23=48$ to get the equation you have just written?

6. If $x+2n=5n$, write what x equals. What must you do to each member of the equation $x+2n=5n$ to get the equation you have just written?

7. If $x+n=45$, write what x equals. What must you do to each member of the equation $x+n=45$ to get the equation you have just written?

8. If $x+n=a$, write what x equals. What must you do to each member of the equation $x+n=a$ to get the equation you have just written?

30. In finding the value of x in the above equations the student has made use of a very evident mathematical truth, which may be stated as follows:

If we take from equals the same number, or equal numbers, the remainders will be equal.

This is one of several equally evident truths which are known in mathematics as **Axioms.**

9. How much more is $x-4$ than $x-5$? If $x-5=3$, what does $x-4$ equal? Why? What does $x-3$ equal? Why? What does $x-2$ equal? Why? What does $x-1$ equal? Why? What does x equal? Why?

10. If $x-7=8$, what does $x-5$ equal? Why? What does $x-3$ equal? Why? What does x equal? Why?

11. If $x-20=25$, what does $x-15$ equal? Why? What does $x-10$ equal? Why? What does $x-4$ equal? Why? What does x equal? Why?

12. If $x-11=13$, write what x equals. What must you do to *each member* of the equation $x-11=13$ to get the equation $x=24$?

13. If $x-16=50$, write what x equals. What must you do to each member of the equation $x-16=50$ to get the equation you have just written?

14. If $x-3n=5n$, write what x equals. What must you do to each member of the equation $x-3n=5n$ to get the equation you have just written?

15. If $x-n=15$, write what x equals. What must you do to each member of the equation $x-n=15$ to get the equation you have just written?

16. If $x-n=a$, write what x equals. What must you do to each member of the equation $x-n=a$ to get the equation you have just written?

31. In finding the value of x in the above equations the student has made use of another very evident mathematical truth, or axiom, which is usually stated as follows:

If we add to equals the same number or equal numbers, the sums will be equal.

EXERCISE 19.

EXAMPLES.

Find the value of x in each of the following equations, by the addition or subtraction of equals:

1. $x-19=32$.

We have $x-19=32$

Adding equals to each member $\underline{19 \quad 19}$

Whence $x=51$

2. $5x+12=87$.

We have $5x+12=87$

Subtracting equals from each member $\underline{12 \quad 12}$

$5x=75$

Whence we know $x=15$.

3. $x-3=17$.
4. $x+17=19$.
5. $x+21=69$.
6. $x-19=43$.
7. $5x+3=28$.
8. $7x+5=26$.
9. $5+6x=29$.
10. $9x-5=31$.

11. $x+140=191$.
12. $18+x=35$.
13. $x-48=56$.
14. $25+x=38$.
15. $2=x-8$.
16. $230=x-103$.
17. $194=11x-26$.
18. $5x+2+x=20$.
19. $4x+5+7x+9-8x=16$.
20. $9x+12-6x-13+2x=19$.
21. $9x+13-x=29$.
22. $13x+9-8x=39$.

EXERCISE 20.

TRANSPOSITION IN EQUATIONS

1. In the following, explain how each equation in the second column can be obtained from the corresponding one in the first column:

(1) $x+4=9$.	(1′) $x=9-4$.
(2) $x+2c=5c$.	(2′) $x=5c-2c$.
(3) $x+2c=12$.	(3′) $x=12-2c$.
(4) $x+a=b$.	(4′) $x=b-a$.
(5) $x-6=3$.	(5′) $x=3+6$.
(6) $x-2d=3d$.	(6′) $x=3d+2d$.
(7) $x-2d=7$.	(7′) $x=7+2d$.
(8) $x-a=b$.	(8′) $x=b+a$.

2. The additive terms $+4$, $+2c$, and $+a$ from the *left* members of the equations in the first column appear as what kind of terms in the *right* members of the equations in the second column?

3. The subtractive terms -6, $-2d$, and $-a$ of the *left* members of the equations in the first column appear as what kind of terms in the *right* members of the equations in the second column?

32. From the above work we learn the following principle:

By the addition or subtraction of equals, we may cause a term to disappear from any member of an equation and to appear with its sign changed in the other member of the equation.

If we remove a term from one member of an equation and make it appear in the other member, we are said to **Transpose** that term. If we use this word we may re-state the above principle as follows:

Any term in one member of an equation may be transposed to the other member provided its sign be changed.

In this way of speaking, we are apt to keep in mind merely the change which results in the equation and to lose sight of the addition or subtraction of equals which *causes* transposition. The axioms must always be appealed to when we are called upon to explain why transposition is allowable.

EXERCISE 21.

EXAMPLES.

33. To **Solve** an equation is to find the value of the unknown number in the equation, and the process of finding this unknown number is called the **Solution** of of the equation.

As mistakes may be made in the solution of an equation, it is well for the student to test the results found, by putting in the original equation the value obtained for the unknown number in place of the letter representing it. If the equation thus found is not true, a mistake has been made and the solution should be re-examined.

This process of testing a result is called the **Verification** of that result.

Solve each of the following equations:

1. $5x-2=3x+18$.

SOLUTION.

Transposing the terms $3x$ and -2,

$$5x-3x=18+2.$$

Uniting terms, $2x=20;$

whence $x=10.$

VERIFICATION.

Substituting 10 for x in the original equation,

$$5\times10-2=3\times10+18,$$

or $50-2=30+18;$

and, since this equation is true, the correct value of x has been found.

2. $30+5x=70.$
3. $9x-5=31.$
4. $7x-8=41.$
5. $15x-13=107.$
6. $28-3y=7.$

SOLUTION.

Transposing the terms $-3y$ and 7,

$$28-7=3y.$$

Uniting terms, $21=3y;$

whence $y=7.$

VERIFICATION.

Substituting 7 for y in the original equation,

$$28-21=7.$$

7. $19-5x=4.$
8. $107-13x=42.$
9. $76-19z=0.$
10. $256-32y=0.$
11. $8x+7-x=14.$
12. $9x+13-x=29.$
13. $17y+19-2y=64.$
14. $23x-18+3x=86.$
15. $3x+18=5x.$
16. $19x-14=12x.$
17. $19x=16+11x.$
18. $7x=100+3x.$
19. $30x=80-10x.$
20. $5x+18=3x+38.$
21. $56x-27=47x.$
22. $9x+17=102-8x.$
23. $5x-5=2x+3.$
24. $7x+2=4x+7.$
25. $3x-1=11-x.$
26. $x+4=10-2x.$
27. $31-7x=41-8x.$
28. $38-2y=9y-39.$
29. $29x-57=16x-5.$
30. $147-19x=122-14x.$
31. $14x+23=19x-2.$
32. $50+17x=295-18x$

33. $5x+13-2x=100-20x-18.$

34. $16x+10-21x=45-10x-15.$

35. $0=14+x-8x-3x+4+x.$

36. $7-5x-10+8x-7+3x=x.$

37. $2x-\frac{3}{5}x=\frac{3}{2}x-\frac{1}{2}-\frac{2}{5}x+2.$

EXERCISE 22.

PROBLEMS.

1. What number increased by 57 is equal to 93?

Let x represent the number.

Then, because the number increased by 57 equals 93, therefore,

$$x+57=93.$$

Transposing, $x=93-57.$

Uniting similar terms, $x=36.$

2. What number diminished by 26 is equal to 29?

3. What number increased by 17 is equal to 35?

4. What number diminished by 19 is equal to 15?

5. If twice a certain number be increased by 12, the result will be 30. What is the number?

Let x represent the number.

Then, because twice the number increased by 12 equals 30, therefore,

$$2x+12=30, \quad \text{etc.}$$

6. If five times a certain number be diminished by 15, the result will be 45. What is the number?

7. Five times a certain number exceeds three times that number by 22. What is the number?

Let x equal the number.

Then, since 5 times the number exceeds 3 times the number by 22, therefore, $5x-3x=22,$ etc.

8. If three times a certain number be added to the number itself, the sum will be 36. What is the number?

9. If five times a certain number be added to eight times that number, the sum will be 78. What is the number?

10. If three times a certain number be increased by 5, the result is equal to the number increased by 25. What is the number?

Let x=the number.
Then $3x+5$=3 times the number increased by 5,
and $x+25$=the number increased by 25.
Then, because 3 times the number increased by 5 equals the number increased by 25, therefore,

$$3x+5=x+25, \quad \text{etc.}$$

11. If three times a certain number be diminished by 4, the result is equal to the number increased by 6. What is the number?

12. What number must be added to 787, so as to obtain the same result as when the number is taken from 875?

13. What number is as much greater than 78 as it is less than 108?

14. What number gives, when doubled, 7 more than three times itself diminished by 16?

15. Having \$78, I spent an amount such that I had left 5 times as much as I spent. How much did I spend?

16. A, B, and C together put \$8781 into a business. B put in \$1000 more than A, and C put in \$2000 more than A. Find how much money each man put in.

Let x=number of dollars A furnished.
Then $x+1000$=number of dollars B furnished,
and $x+2000$=number of dollars C furnished,
and since altogether they furnished \$8781, therefore,

$$x+(x+1000)+(x+2000)=8781.$$

Removing parentheses and uniting similar terms

$$3x+3000=8781, \quad \text{etc.}$$

17. A boat went 368 miles in three days. The second day it sailed 32 miles further than it did the first day, but the third day it sailed 24 miles less than it did the first day. How far did the boat sail each day?

18. John, Fred, and Dave have $4.35. Fred has twice as much as John, and Dave has 75 cents more than John. How much has each?

19. A man dying left his property worth $52800 to his four children. He gave his oldest child $3000 more than the youngest, the next $2000 more than the youngest, and the next $1000 more than the youngest. How much did each receive?

20. A merchant made $4222 in three years. He made $1592 more the second year than he did the first, but the third he lost all he made the first year and $350 more. What did he make each year?

21. If 63 be added to a certain number, the number becomes 10 times as large. What is the number?

22. A and B had equal sums of money. A doubled his money and then made $25, while B tripled his money and then lost $100. They then had equal amounts. What sum did each have at first?

Let x = the number of dollars each had at first.

Then, because A doubled his money, and also made $25, therefore,

$2x + 25$ = the number of dollars A had finally.

Also, because B tripled his money and also lost $100, therefore,

$3x - 100$ = the number of dollars B had finally.

Now, since A and B finally had equal amounts

$3x - 100 = 2x + 25$, etc.

23. If 10 times a certain number be diminished by 22, there results the same as when 7 times the number is increased by 23. What is this number?

24. A man doubled the money he had, and then made $500. He then made an amount equal to 3 times what he had at first, but losing $5400 he had nothing left. How much did he have at first?

25. A man walked 47 miles in three days. He walked 8 miles more the second day than he did the first, and 10 miles more the third day than he did the second. Find how far he walked each day.

26. A farmer rode in a carriage from his home to the railroad, and then he rode on the cars 5 times as far as in the carriage, and then on a steamboat 8 times as far as in the carriage and on the cars together, when he had traveled in all 270 miles. How far did he live from the railroad?

CHAPTER V.

MULTIPLICATION.

EXERCISE 23.

GENERAL DEFINITION OF MULTIPLICATION.

34. The original meaning of multiplication in Arithmetic is that of *repeated addition*, and, with this meaning in mind, we would define multiplication to be the taking of one number as many times as there are units in another. Thus, 3 multiplied by 5 means $3+3+3+3+3$, and $\frac{2}{3}$ multiplied by 5 means $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$. As soon, however, as the multiplier is a fraction, it is found that this meaning of multiplication does not apply; for while 3 can be repeated 5 times, yet 5 *cannot* be repeated $\frac{1}{2}$ a time, nor can $\frac{2}{3}$ be repeated $\frac{4}{5}$ a time. Now, although the operation of multiplying $\frac{2}{3}$ by $\frac{4}{5}$ cannot be looked upon as repeated addition, yet this operation does occur in Arithmetic, and is called multiplication. It is plain, therefore, that the word is used with some other meaning than that originally given it, which new meaning may be stated as follows: *

"*To multiply one number by another, we do to the first what is done to unity to obtain the second.*"

Thus, suppose we are required to multiply 3 by 5. To make 5 from unity, we must take unity 5 times, and hence to multiply 3 by 5 we must take *three*, 5 times

Again, suppose we wish to multiply $\frac{2}{3}$ by 5. To make 5 from unity, we must take unity 5 times, and hence to multiply $\frac{2}{3}$ by 5 we must take *two-thirds*, 5 times.

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*Boset *Algebra Elementaire*, Charles Smith's Elementary Algebra.

Suppose we are required to multiply 5 by $\frac{3}{4}$. To make $\frac{3}{4}$ from unity, we must divide unity into 4 equal parts and take the result 3 times. Hence to multiply 5 by $\frac{3}{4}$ we must divide 5 into 4 equal parts giving $\frac{5}{4}$ and take this result 3 times; that is

5 multiplied by $\frac{3}{4}$ is $\frac{5}{4}\times 3$ or $\frac{15}{4}$.

Finally, suppose we wish to multiply $\frac{2}{3}$ by $\frac{4}{5}$. To make $\frac{4}{5}$ from unity, we must divide unity into 5 equal parts giving $\frac{1}{5}$ and take this result 4 times. Hence to multiply $\frac{2}{3}$ by $\frac{4}{5}$ we must divide $\frac{2}{3}$ into 5 equal parts giving $\frac{2}{3\times 5}$ or $\frac{2}{15}$ and take this result four times; that is

$\frac{2}{3}$ multiplied by $\frac{4}{5}$ is $\frac{2}{3\times 5}\times 4$ or $\frac{8}{15}$.

1. What must you do to unity to produce 8? Explain, then, how 5 is multiplied by 8. Explain how $\frac{5}{6}$ is multiplied by 8.

2. What must you do to unity to produce $\frac{1}{8}$? Explain, then, how 7 is multiplied by $\frac{1}{8}$. Explain how $\frac{2}{5}$ is multiplied by $\frac{1}{8}$.

3. What must you do to unity to produce $\frac{3}{7}$? Explain, then, how 9 is multiplied by $\frac{3}{7}$. Explain how $\frac{2}{5}$ is multiplied by $\frac{3}{7}$.

Explain, by the general definition of multiplication, how the product is found in each of the following cases:

4. 11 multiplied by 12.
5. 11 multiplied by 1.2
6. 3 multiplied by $\frac{7}{8}$.
7. $\frac{2}{9}$ multiplied by $\frac{7}{8}$.
8. $\frac{7}{8}$ multiplied by 3.
9. $\frac{7}{8}$ multiplied by $\frac{2}{9}$.
10. $\frac{2}{3}$ multiplied by $\frac{3}{5}$.
11. $\frac{3}{5}$ multiplied by $\frac{4}{6}$.
12. 4 multiplied by $\frac{1}{2}$.
13. 6 multiplied by 6.
14. $\frac{3}{5}$ multiplied by $\frac{3}{5}$.
15. $\frac{3}{5}$ multiplied by $\frac{5}{3}$.
16. 15 multiplied by $\frac{3}{5}$.
17. 1.6 multiplied by .25
18. $12\frac{1}{2}$ multiplied by $\frac{3}{4}$.
19. $\frac{3}{2}$ multiplied by $12\frac{1}{2}$.
20. 100 multiplied by .01
21. .001 multiplied by .01

EXERCISE 24.

MULTIPLICATION OF MONOMIALS.

35. If an expression consists of but one term it is called a **Monomial**, and if it consists of more than one term it is called a **Polynominal.** Thus, $6ab$ is a monomial and $3a^2-4b+2$ is a polynominal.

If a polynomial consists of just two terms, it is called a **Binomial**, and if it consists of just three terms, it is called a **Trinomial.** Thus, $3a-4x^2$ is a binomial and $6x-4xy+8$ is a trinomial.

While binomials and trinomials are each polynomials, yet it is usual to apply the word polynomial only to expressions of more than three terms.

36. It is one of the laws of multiplication, discovered in Arithmetic, that the product will be the same, no matter in what order the factors are multiplied together, or as is briefly stated:

Multiplication may be performed in any order.

This is called the **Commutative Law** of multiplication, and may be illustrated as follows:

$$2\times5\times7\times12=12\times7\times2\times5=7\times2\times5\times12, \quad \text{etc.}$$

$$\tfrac{2}{3}\times5\times\tfrac{4}{5}=\tfrac{4}{5}\times\tfrac{2}{3}\times5=5\times\tfrac{4}{5}\times\tfrac{2}{3}=\tfrac{4}{5}\times5\times\tfrac{2}{3}, \quad \text{etc.}$$

And if a, b, and c stand for *any numbers whatever*, we may say,

$$abc=bca=cab=acb=bac=cba.$$

1. How many times 6 is 5 times twice 6? How many times a certain number is 5 times twice that number? How many times n is $5\times2n$?
2. How many times x is $9\times8x$?
3. How much is $8\times11a$?
4. How much is $25\times4b$?

5. How much is $15 \times 12xy$?

6. How many times 2×3 is $2 \times 5 \times 3$? How many times ab is $a \times 5b$? How many times xy is $x \times 8y$?

7. How much is $x^3 \times 31y^2$?

8. How many times 2×3 is $5 \times 2 \times 7 \times 3$? How many times ab is $3a \times 8b$? How many times xy is $7x \times 9y$?

9. How much is $9x \times 11y^2$?

10. How many times abc is $9c \times 10ab$?

11. How many times $abxy$ is $21ay \times 15bx$?

37. From the above work we learn that:

The product of two monomials is found by multiplying the numerical coefficients of the monomials to obtain the numerical coefficient of the product, and by writing in succession the literal factors of the monomials for the literal part of the product.

EXERCISE 25.

EXAMPLES.

Multiply	Multiply
1. $7ax$ by 9.	11. 14 by $\frac{1}{7}ay$.
2. $8a^5x$ by 11.	12. $\frac{3}{10}$ by $\frac{5}{6}ax^3$.
3. $\frac{1}{2}gt^2$ by 234.	13. mn by dy.
4. $8c^2x$ by 80.	14. xy by abz.
5. $10a^2c^2$ by 13.	15. ab^2 by c^3y.
6. $12bc^3$ by $\frac{2}{3}$.	16. d^2m by $5ax$.
7. 25 by $2x$.	17. mn^2 by $9y^2$.
8. 11 by $4ab$.	18. $7a^2x$ by dy.
9. 90 by $6ax^2$.	19. $10a^2f$ by c^2d.
10. 15 by $3mn$.	20. d^2n by $9av$.

	Multiply		Multiply
21.	$12a^2c$ by $4m$.	28.	$\frac{7}{8}cx^5$ by $\frac{1}{3}b^2y$.
22.	$11dy^2$ by $8gx^2$.	29.	$\frac{3}{4}a^3m^3$ by $\frac{2}{3}cx$.
23.	$6m^3p^2$ by $7an^5$.	30.	$\frac{5}{6}b^2n^3$ by $\frac{3}{10}c^3m$.
24.	$15ax^4$ by $3b^4u$.	31.	$\frac{2}{3}c^2d$ by $\frac{9}{10}mx^2$.
25.	$\frac{3}{4}c^2x^3$ by $\frac{3}{5}a^3b^2$.	32.	$1\frac{1}{3}dx^2$ by $\frac{3}{4}\frac{9}{0}y^5b^2$.
26.	$\frac{2}{9}m^2n^3$ by $\frac{4}{5}d^3y$.	33.	$4.6c^2d^5$ by $.3a^2b$.
27.	$\frac{5}{6}b^2y$ by $\frac{7}{8}a^3y^3$.	34.	$.5a^2x$ by $.6b^3y$.

EXERCISE 26.

LAW OF EXPONENTS IN MULTIPLICATION.

1. What is the product of aaa and aa, written in the *abbreviated form?* What, then, is the product of a^3 and a^2?

2. What is the product of ccc and $cccc$, written in the *abbreviated form?* What, then, is the product of c^3 and c^4?

3. In y^3, how many times is y used as a factor? In y^4, how many times is y used as a factor? In y^3 *times* y^4, how many times is y used as a factor? How, then, would you write the product $y^3 \times y^4$ in the simplest form?

4. In a^5, how many times is a used as a factor? In a^3, how many times is a used as a factor? In a^5 *times* a^3, how many times is a used as a factor? Write $a^5 \times a^3$ in the simplest form.

How is the exponent in this simplest form obtained from the exponents 5 and 3?

5. In a^5, how many times is a used as a factor? If n stands for a certain whole number, how many times is a used as a factor in a^n? In a^5 *times* a^n, how many times is a used as a factor?

How, then, would the exponent of the product be formed from the exponents 5 and n?

6. If n stands for a certain whole number, how many times is a used as a factor in a^n? If r stands for some other whole number, how many times is a used as a factor in a^r? In a^n *times* a^r, how many times is a used as a factor?

How, then, would the exponent of the product of any two powers of a be found from the exponents of the factors?

38. From the above we learn the **Law of Exponents in Multiplication,** which is usually stated as follows:

The product of two powers of the same number is equal to that number with an exponent equal to the sum of the exponents of the two factors.

39. In mathematics statements like the above are often expressed in the symbolic language of Algebra, and when thus expressed are called **Formulas.** The above law expressed as a *formula* would be,

$$a^n a^r = a^{n+r}.$$

Since a is *any number* and n and r are *any whole numbers,* this algebraic equation is equivalent to saying:

The product of two powers of the same number is equal to that number with an exponent equal to the sum of the exponents of the two factors.

EXERCISE 27.

EXAMPLES.

Multiply

1. $3a^2$ by $11a^3$.
2. $5x^3$ by $15x^2$.
3. $14b^2$ by cb^4.
4. $14cb^2$ by b^4.

Multiply

5. $5a^2b^2$ by $9ab$.
6. $7ab^2$ by $8a^2b^2$
7. $11xy^2$ by $3x^2y^3$.
8. axy by xyz.

Multiply	Multiply
9. abc by bcd.	18. $20b^2z$ by $.3a^2b$.
10. $8ax^3$ by $5bx^5$.	19. $5mx$ by $1\frac{1}{5}b^2x^5$.
11. $12aby^3$ by $10acy^2$.	20. $5x^2z^3$ by $20nx^3$.
12. $\frac{5}{6}a^2x$ by $4ax^2$.	21. $4mn^2$ by $\frac{5}{6}m^2z^4$.
13. $\frac{1}{2}a^2y^5$ by $4a^4y^4$.	22. $2a^4y^3$ by ay^2.
14. $3a^2x^3$ by $3a^5m^2x^2$.	23. $2\frac{1}{11}bx$ by $5\frac{1}{2}b^2y^3$.
15. $6p^2x^5$ by $5b^4p^3x^2$.	24. $33ac$ by $10\frac{1}{3}a^2y^4$.
16. $8b^2$ by $\frac{1}{2}a^3b^3$.	25. $(x+y)^3$ by $4(x+y)^2$.
17. $60xy^2$ by $.05x^3y$.	26. $3(x-5)^2$ by $a(x-5)^4$.

EXERCISE 28.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS.

1. How much more than 200 is 2 times $(100+12)$? How much more than 300 is 3 times $(100+12)$? How much more than 500 is 5 times $(100+12)$? How much more than n hundred is n times $(100+12)$?

2. How much more than 200 is 2 times $(100+b)$? Write, then, what $2\times(100+b)$ equals.

3. How much more than 600 is 6 times $(100+c)$? Write, then, what $6\times(100+c)$ equals.

4. How much more than $2a$ is 2 times $(a+12)$? Write, then, what $2\times(a+12)$ equals.

5. How much more than $4a$ is 4 times $(a+9)$? Write, then, what $4\times(a+9)$ equals.

6. How much more than na is n times $(a+9)$? Write, then, what $n\times(a+9)$ equals.

7. How much more than na is n times $(a+15)$? Write, then, what $n\times(a+15)$ equals.

8. How much more than na is n times $(a+b)$? Write, then, what $n \times (a+b)$ equals.

9. How much less than 200 is 2 times $(100-12)$? How much less than 300 is 3 times $(100-12)$? How much less than 500 is 5 times $(100-12)$? How much less than n hundred is n times $(100-12)$?

10. How much less than 200 is 2 times $(100-b)$? Write, then, what $2 \times (100-b)$ equals.

11. How much less than 600 is 6 times $(100-c)$? Write, then, what $6 \times (100-c)$ equals.

12. How much less than $2a$ is 2 times $(a-12)$? Write, then, what $2 \times (a-12)$ equals.

13. How much less than $4a$ is 4 times $(a-9)$? Write, then, what $4 \times (a-9)$ equals.

14. How much less than na is n times $(a-9)$? Write, then, what $n \times (a-9)$ equals.

15. How much less than na is n times $(a-15)$? Write, then, what $n \times (a-15)$ equals.

16. How much less than na is n times $(a-b)$? Write, then, what $n \times (a-b)$ equals.

40. The principles we learn above may be stated as follows:

The product of the sum of two numbers by a third number equals the sum of the products of each of the two numbers by the third number.

The product of the difference of two numbers by a third number equals the difference of the products of each of the two numbers by the third number.

These principles may be stated in algebraic language, by means of the following formulas:

$$n(a+b)=na+nb.$$
$$n(a-b)=na-nb.$$

We will now proceed to the case where the multiplicand has more than two terms.

17. How much more than $5(a+b)$ is $5(a+b+3)$? How much more than $5(a+b)$ is $5(a+b+c)$? Write an expression equal to $5(a+b+c)$ without using a parenthesis.

18. How much less than $5(a+b)$ is $5(a+b-3)$? How much less than $5(a+b)$ is $5(a+b-c)$? Write an expression equal to $5(a+b-c)$ without using a parenthesis.

19. How much more than $7(a-b)$ is $7(a-b+4)$? How much more than $7(a-b)$ is $7(a-b+c)$?

Write an expression equal to $7(a-b+c)$ without using a parenthesis.

20. How much less than $7(a-b)$ is $7(a-b-4)$? How much less than $7(a-b)$ is $7(a-b-c)$?

Write an expression equal to $7(a-b-c)$ without using a parenthesis.

21. How much more than $n(a+b)$ is $n(a+b+c)$? How much more than $n(a-b)$ is $n(a-b+c)$?

Write what $n(a+b+c)$ and $n(a-b+c)$ equal without using parentheses.

22. How much less than $n(a+b)$ is $n(a+b-c)$? How much less than $n(a-b)$ is $n(a-b-c)$?

Write what $n(a+b-c)$ and $n(a-b-c)$ equal without using parentheses.

41. What we have learned may be expressed by formulas as follows:

$$n(a+b+c)=na+nb+nc,$$
$$n(a-b+c)=na-nb+nc,$$
$$n(a+b-c)=na+nb-nc,$$
$$n(a-b-c)=na-nb-nc,$$

and, by a continuation of the questions above given, it is easy to see that we would find,

$$n(a+b-c+d)=na+nb-nc+nd,$$
$$n(a-b+c-d)=na-nb+nc-nd,$$
$$n(a-b+c+d-e)=na-nb+nc+nd-ne,$$

and so on. Embodying all these in a single statement, we can say:

The product of a polynomial by a monomial is the aggregate obtained by placing the multiplier as a factor in each term of the polynomial.

EXERCISE 29.

EXAMPLES.

Write the product in each of the following examples without using a parenthesis:

1. $7(2x-4y+z)$.
2. $3a(4g+2h-5k)$.
3. $2x(14a-6b-3c)$.
4. $6y(5n-6r+4t)$.
5. $2x(x-4x^2+6)$.
6. $4ax(3+4x-9x^2)$.
7. $7x(a-2b+3c-d)$.
8. $3a(x-4y^2+2z-8n^2)$.
9. $ab(nr+r-n-6)$.
10. $2arm(f-a+r-m)$.
11. $7x(3-x+2x^2-9x^3)$.
12. $9ay(11-4y+2y^2+5y^3)$.

EXERCISE 30.

ARRANGEMENT OF WORK IN THE MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL.

42. The following illustrations will be sufficient to explain the usual arrangement of work when the product of a polynomial by a monomial is sought:

Suppose it is required to multiply $a+b-c$ by 5.

Multiplicand,	$a+\ b-\ c$
Multiplier,	5
Product,	$5a+5b-5c$

Suppose it is required to multiply $x-3x^2+2x^3$ by $5x$.

Multiplicand,	$x - 3x^2 + 2x^3$
Multiplier,	$5x$
Product,	$5x^2-15x^3+10x^4$

Suppose it is required to multiply $6ay^2$ by $2a^2y^2-3ay+5$. Since *multiplication may be performed in any order*, we arrange this just as though $6ay^2$ were the multiplier.

$$\begin{array}{r} 2a^2y^2 - 3a\,y + 5 \\ 6a\,y^2 \\ \hline 12a^3y^4-18a^2y^3+30ay^2 \end{array}$$

This arrangement of work is analogous to the arrangement of similar work in Arithmetic, as the following example will show:

		Hundreds.	Tens.	Units.
Multiplicand,	$7h+2t+3u$	7	2	3
Multiplier,	3			3
Product,	$21h+6t+9u$	21	6	9

EXAMPLES.

1. Multiply $x^2+2xy+z$ by 7

2. Multiply $2x-5y+3$ by 11

3. Multiply x^2+2x+3 by $3x$

4. Multiply $5a^2-4a-6$ by $4ax$

Multiply

5. $2x-4y+z$ by $3x$.
6. $a^2-2ab-b^2$ by a^3b^2.
7. a^2d by $4a+2ac^3-3d^4-1$.

Remember that it is indifferent which expression is used as the multiplier.

8. $3yx^2$ by x^3+4x^2-9x+6.
9. $ay^3+3by^2-2cy-12d$ by $10ny^3$.
10. $6a^2n$ by $3n^2-7a^2+4n-2a$.

Multiply

11. $6ax^2-3a^2x+10ax$ by $5ax^2$.

12. $12x^2y^3$ by $5y+6xy-12x^2y^2$.

13. $x^3-3x^2y+3xy^2-2$ by $3xy^2$.

14. $4ab$ by $2a^2b^2+5ab^3-7b^4$.

15. $2a^2b-3cd^3+\frac{1}{2}ac^3-5$ by $6ac^2d^2$.

16. $3ab+4a^2b-5cb+6$ by $4a^4d^2$.

17. $6a^4-4b^2+2ab^3-3c^2$ by $\frac{1}{2}a^2bc$.

18. $6x^3-8x^2y+2xy^2$ by x^2y^3.

19. $a-\frac{1}{2}b+\frac{1}{3}c-\frac{1}{4}d$ by $24abcd$.

20. $3b^3+9b^2c-b^2-27$ by $\frac{2}{3}b^2c$.

Simplify each of the following expressions:

21. $4(4a-7b)+7(2a-5b)$.

$4(4a-7b)=16a-28b$; $\quad 7(2a-5b)=14a-35b$.

Therefore, $4(4a-7b)+7(2a-5b)=(16a-28b)+(14a-35b)$.

Removing parentheses and combining similar terms,

$4(4a-b)+7(2a-5b)=30a-63b$.

22. $4(4a-7b)-7(2a-7b)$.

$4(4a-7b)=16a-28b$; $\quad 7(2a-5b)=14a-35b$.

Therefore, $4(4a-7b)-7(2a-5b)=(16a-28b)-(14a-35b)$.

Removing parentheses, $=16a-28b-14a+35b$.

Combining similar terms, $=2a+7b$.

23. $5(x+5)-2(x-4)+3(2x-1)$.

24. $3(7a-4b)-4(5a+2b)-2(b-3a)$.

25. $a(a+b-c)-b(a-b+c)+c(a+b)$.

26. $7x(3x+4y-6)-4y(7x-3y+14)$.

27. $14(3a+4b)-6(2a-6b)-4x(x^2-1)$.

28. $6a^2(a^2-2ax)-4y(y+4xy)+8(a^2-ay)$.

29. $n(a-b+c)+r(a-b+c)-t(a-b+c)$.

30. $3ab(a-c)-bc(2b-3a)+6ab^2-b^2(3a-2c)$.

31. $3x(a+b-2y)-2y(a-3b-3x)-3b(x+2y)+2ay$.

32. $6ab(ab^2-4cd)abc$.

$$6ab \times abc = 6a^2b^2c.$$

Therefore, $$6ab(ab^2-4cd)abc = 6a^2b^2c(ab^2-4cd)$$
$$= 6a^3b^4c - 24a^2b^2c^2d.$$

33. $3(16-8x-4x^2+2x^3)6x^2$.

34. $3a(2a^2+3a^2b-3ab^2-4b^2)3b$.

35. $ab^2(ax-bx+6ax^3-5bx^4)4a^2b$.

36. $\frac{1}{4}xy^2(3-xy^2+9x^2y-7y^3)8x^2$.

37. $2a^2b(ax^2-bx+b^2)ab^2$.

EXERCISE 31.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS.

43. Just as it is customary to write $7(10-3+5)$ instead of $7\times(10-3+5)$, so usually the product of two polynomials is indicated by writing it like

$$(14+2-9)(10-3+5),$$

instead of using the sign $\times$, as in $(14+2-9)\times(10-3+5)$. So the student must remember that it is the *product* of two polynomials which is called for when they are enclosed in parentheses with *no sign* between them.

1. If $a+b+c$ multiplied by n is $an+bn+cn$, what is $a+b+c$ multiplied by $(n+r)$?

Write $a(n+r)+b(n+r)+c(n+r)$ without using parentheses.

2. If $a-b+c$ multiplied by n is $an-bn+cn$, what is $a-b+c$ multiplied by $(n+r)$?

Write $a(n+r)-b(n+r)+c(n+r)$ without using parentheses.

3. If $a+b-c$ multiplied by n is $an+bn-cn$, what is $a+b-c$ multiplied by $(n-r+t)$?

Write $a(n-r+t)+b(n-r+t)-c(n-r+t)$ without using parentheses.

4. Could results similar to those above, be obtained *no matter what polynomial is used as a multiplier?*

44. As the conclusion from the above, we have the following principle:

The product of two polynomials is the aggregate obtained by placing one polynomial as a factor in each term of the other polynomial.

EXERCISE 32.

EXAMPLES.

Find the product in each of the following:

1. $(x-4)\ (x+9)$.

Placing the second polynomial as a factor in each term of the first polynomial, we have $x(x+9)-4(x+9)$.

Multiplying by x and 4, we obtain

$$(x^2+9x)-(4x+36).$$

Removing parentheses, we get

$$x^2+9x-4x-36.$$

Uniting similar terms, we have

$$x^2+5x-36$$

which is the required product.

2. $(3a^2+2a-9)(2ab-6b)$.

Placing the second polynomial as a factor in each term of the first polynomial, we have

$$3a^2(2ab-6b)+2a(2ab-6b)-9(2ab-6b).$$

Multiplying by $3a^2$, $2a$ and 9, we obtain

$$(6a^3b-18a^2b)+(4a^2b-12ab)-(18ab-54b).$$

Removing parentheses, we get

$$6a^3b-18a^2b+4a^2b-12ab-18ab+54b.$$

Uniting similar terms, we have

$$6a^3b-14a^2b-30ab+54b$$

which is the required product.

3. $(x+3)(x+7)$.

4. $(x+3)(x-7)$.

5. $(x-3)(x+7)$.

6. $(x-3)(x-7)$.

7. $(x+5)(x-5)$.

8. $(x+6)(x+6)$.

9. $(x+8)(x-4)$.
10. $(x-5)(x-5)$.
11. $(x+12)(x-1)$.
12. $(x-12)(x-1)$.
13. $(x+15)(x-15)$.
14. $(x-17)(x-18)$.
15. $(3x-5y)(3x+5y)$.
16. $(3x-5y)(3x-5y)$.
17. $(3x-5y)(5x-3y)$.
18. $(2a-2b)(2a-b)$.
19. $(4x+9y)(x-5y)$.
20. $(4b-5c)(3b+4c)$.
21. $(3a-4b)(a-b)$.
22. $(5x+1)(7y-2)$.
23. $(a-5m)(a+3m)$.
24. $(3a+5x)(7b-4x)$.
25. $(2a-3b)(5x-7y)$.
26. $(m-n)(x+y)$.
27. $(a+b)(c-d)$.
28. $(2x^2-4x+9)(x-7)$.
29. $(3x^2+2x-6)(2x-3)$.
30. $(x^2-xy+y^2)(x-y)$.
31. $(3x^2-2xy+6y^2)(2x+3y)$.
32. $(a^2-2ab+b^2)(a-b)$.
33. $(3x^2-4x+7)(5x^2-x-4)$.
34. $(x^2+7x-5)(x^2-3x+7)$.
35. $(3a^2-5ab+2b^2)(a^2-7ab)$.
36. $(2xy-y-x)(x-3+y)$.

EXERCISE 33.

ARRANGEMENT OF WORK IN THE MULTIPLICATION OF TWO POLYNOMIALS.

45. Let us go through the work of multiplying the two polynomials $7x^2-6x-9$ and $3x^2-5x+2$ together. We first place the second polynomial as a factor in each term of the first polynomial, and obtain

$$7x^2(3x^2-5x+2)-6x(3x^2-5x+2) \\ -9(3x^2-5x+2). \quad (1)$$

Then multiplying the expressions in the parentheses by $7x^2$, $6x$, and 9 respectively, we obtain

$$(21x^4-35x^3+14x^2)-(18x^3-30x^2+12x) \\ -(27x^2-45x+18). \quad (2)$$

Removing the parentheses, we have

$$21x^4-35x^3+14x^2-18x^3+30x^2-12x \\ -27x^2+45x-18. \quad (3)$$

Now, uniting similar terms, we get

$$21x^4-53x^3+17x^2+33x-18 \quad (4)$$

which is the required product.

This same work can be arranged in a convenient form, as follows:

Multiplicand,	$3x^2-5x+2$	(5)
Multiplier,	$7x^2-6x-9$	(6)
1st partial product,	$(21x^4-35x^3+14x^2)$	(7)
2d partial product,	$-(18x^3-30x^2+12x)$	(8)
3d partial product,	$-(27x^2-45x+18)$	(9)

where the parentheses are the same as in (2) above, the only difference being that they are arranged so as to bring *similar terms in the same vertical column*. Removing these parentheses and uniting the similar terms in the same column, we have

Multiplicand,	$3x^2-5x+2$	(10)
Multiplier,	$7x^2-6x-9$	(11)
1st partial product,	$21x^4-35x^3+14x^2$	(12)
2d partial product,	$-18x^3+30x^2-12x$	(13)
3d partial product,	$-27x^2+45x-18$	(14)
Product,	$21x^4-53x^3+17x^2+33x-18$	(15)

which is the usual arrangement of work in the multiplication of two polynomials.

46. The expressions (12), (13), and (14), which we have called *partial products*, were obtained, as we have just stated, in the following manner:

First. The expression $3x^2-5x+2$ *was placed as a factor in each term of the expression* $7x^2-6x-9$.

Second. The multiplications by $7x^2$, $6x$, *and* 9 *were performed.*

Third. The parentheses were removed.

Hence it follows that when the multiplicand is placed as a factor in the *additive* term $7x^2$, the result is the partial product (12), which has its terms

additive, *subtractive*, *additive*,

the *same* as the multiplicand. But when the multiplicand is placed as a factor in the *subtractive* terms $-6x$ and -9, the results are the partial products (13) and (14), which have their terms

subtractive, *additive*, *subtractive*,

which are just the *opposite* of those of the multiplicand.

47. From this reasoning we can readily formulate the following method for finding the product of any two given polynomials:

Neglect the signs of the terms, and multiply, in succession, the multiplicand by each term of the multiplier, to obtain the successive partial products.

Give the terms of each partial product, the same signs as those of the multiplicand when an additive term of the multiplier is used, but give the terms of each partial product just the opposite signs to those of the multiplicand when a subtractive term of the multiplier is used.

Add the partial products thus formed, and the result is the required product.

Thus, suppose $(x^2+3x-4)(x^2-5x+2)$ is required,

$$\begin{array}{r} x^2+3x-4 \\ x^2-5x+2 \\ \hline x^4+3x^3-4x^2 \\ -5x^3-15x^2+20x \\ 2x^2+6x-8 \\ \hline x^4-2x^3-17x^2+26x-8 \end{array}$$

The above way of obtaining the signs of the terms in the partial products keeps the "*reasons why*" prominently before the student, but in practice the signs are more often determined in another manner, which we proceed to explain.

48. In the above we have seen that the signs in the partial products will be the same as those of the multiplicand if the term used in the multiplier is additive. Hence we have the two following cases:

If a term in multiplicand is *additive*, *i. e.*, has sign + and term used in multiplier is *additive*, *i. e.*, has sign + the resulting term in product is *additive*, *i. e.*, has sign +

If a term in multiplicand is *subtractive*, *i. e.*, has sign − and term used in multiplier is *additive*, *i. e.*, has sign + the resulting term in product is *subtractive*, *i. e.*, has sign −

We have also learned that the signs in the partial products will be the opposite to those of the multiplicand if the term used in the multiplier is subtractive. Hence we have the two following cases:

If a term in multiplicand is *additive*, *i. e.*, has sign + and term used in multiplier is *subtractive*, *i. e.*, has sign − the resulting term in product is *subtractive*, *i. e.*, has sign −

If a term in multiplicand is *subtractive*, *i. e.*, has sign − and term used in multiplier is *subtractive*, *i. e.*, has sign − the resulting term in product is *additive*, *i. e.*, has sign +

From this we see that when the terms in multiplicand and multiplier are *both additive* or *both subtractive*, the resulting term in the product is *additive*; but when the terms in multiplicand and multiplier are *one additive* and *one subtractive*, the resulting term in the product is *subtractive*. From this, results the statement known as the **Rule of Signs:**

In multiplication, like signs give plus and unlike signs give minus.

49. Now the method given in the previous article may be stated as follows:

To Multiply two Polynomials together, *neglect the signs of the terms and multiply, in succession, the multiplicand by each term of the multiplier, to obtain the successive partial products.*

Determine the sign of each term of the partial products by the Rule of Signs.

Add the partial products thus formed and the result is the required product.

50. The following examples will tend to show the advantage of arranging the work of multiplication in the way explained:

$$\begin{array}{r} x + y \\ x + y \\ \hline x^2 + xy \\ \quad xy + y^2 \\ \hline x^2 + 2xy + y^2 \end{array} \qquad \begin{array}{r} a - b \\ a + b \\ \hline a^2 - ab \\ \quad ab - b^2 \\ \hline a^2 \quad - b^2 \end{array} \qquad \begin{array}{r} x^2 + x\,y + y^2 \\ x - y \\ \hline x^3 + x^2y + xy^2 \\ \quad - x^2y - xy^2 - y^3 \\ \hline x^3 \quad\quad - y^3 \end{array}$$

"The student should observe that, with the view of readily bringing similar terms of the product into the same column, the terms of the multiplicand and multiplier are arranged in a certain order. We fix on some letter which occurs in many of the terms, and arrange the terms *according to the powers of that letter.* Thus, taking the last example, we fix on the letter x; we put first in the multiplicand the term x^2, which contains the highest power of x, namely the second power; next we put the term xy which contains the next power of x, namely the first; and last we put the term y^2 which does not contain x at all. The multiplicand is then said to be arranged *according to descending powers of* x. We arrange the multiplier in the same way.

We might also have arranged both multiplicand and multiplier in reverse order, in which case they would be arranged *according to ascending powers of* x. It is of no consequence which order we adopt, but we should take the *same* order for the multiplicand and the multiplier," — *Todhunter's Algebra for Beginners.*

EXERCISE 34.

EXAMPLES.

Multiply the following expressions:

1. $x-13$ by $x-14$.
2. $x+19$ by $x-20$.
3. $2x-3$ by $x+8$.
4. $2x+3$ by $x-8$.
5. $x-5$ by $2x-1$.
6. $2x-5$ by $x-1$.
7. $3x-5$ by $2x+8$.
8. $5x+6$ by $2x-3$.
9. $3a+7$ by $5a+9$.
10. $7a+4$ by $6a^2+3a$.
11. $2xy+y^2$ by x^2-2xy.
12. $3a^2-4a$ by $2a^3+6a^2$.
13. $3x^2+2y^2$ by $3x^2-2y^2$.
14. x^2-xy+y^2 by $x+y$.
15. $5a^2+a+8$ by $5a+6$.
16. $5+2a+8a^2$ by $6+8a+3a^2$.
17. $2a+9a^2+3a^3$ by $8+4a+5a^2$.
18. $3-2x+x^2$ by $5-x$.
19. $x^2-2ax+5a^2$ by $x-3a$.
20. $7-4x-3x^2+5x^3$ by $2+5x$.
21. $2u^2-7uv-3v^2$ by $3u-2v$.
22. $5a^2-3ax-5x^2$ by $3a-5x$.
23. $1-2x+3x^2$ by $2+3x-4x^2$.
24. $2x^2-x+3$ by $2x^2-2x-4$.
25. $3a^2-2ax+7x^2$ by $3a^2+2ax-7x^2$.
26. $7a^2+2ax-4x^2$ by $7a^2-2ax-4x^2$.
27. $5a^3-2a^2x+ax^2$ by $2a^2-ax+2x^2$.
28. $2x^2-3+5x^3$ by $6x-8+4x^3$.

Arranging according to the descending powers of x, we obtain

$$
\begin{array}{r}
5x^3+2x^2-3 \\
4x^3+6x\ -8 \\
\hline
20x^6+8x^5\qquad -12x^3 \\
+30x^4+12x^3\qquad -18x \\
-40x^3-16x^2\qquad +24 \\
\hline
20x^6+8x^5+30x^4-40x^3-16x^2-18x+24
\end{array}
$$

29. $7-4x+3x^2$ by $5x^2-x-4$.

30. $7x-5+x^2$ by $7-3x+x^2$.

31. $ab^3+3a^3b-2a^2b^2$ by $2a^2-ab-5b^2$

32. x^3-2x+1 by x^3-3x+2.

33. $x^2-5ax-2a^2$ by $x^2+2ax+3a^2$.

34. $7x^2+y^2-3xy$ by $2x^3+y-x$.

35. $2x^4-4x^2-4x-1$ by $2x^4-4x^2-4x-1$.

36. $5x-7x^2+x^3+1$ by $1+2x^2-4x$.

37. $11-4x+x^2$ by $4x^2-24+5x+3x$.

38. $4x^2-3xy-y^2$ by $3x-2y$.

39. $a^2-ax+bx+b^2$ by $a+b+x$.

40. $x^2+y^2-xy+x+y-1$ by $x+y-1$.

41. $x+2y-3z$ by $x-2y+3z$.

42. $a^2+b^2+c^2-bc-ca-ab$ by $a+b+c$.

43. $a^2-2ab+b^2+x^2$ by $a^2+2ab+b^2-x^2$.

44. $3a^2+2a+2a^3+1+a^4$ by a^2-2a+1.

45. $x^2+y^2+z^2+xy-yz+xz$ by $x-y+z$.

46. $\frac{1}{2}x^2-2x+\frac{3}{2}$ by $\frac{1}{2}x+\frac{1}{3}$.

47. $\frac{3}{2}a+\frac{2}{3}b$ by $\frac{2}{3}a-\frac{3}{2}b$.

48. $\frac{5}{2}x-\frac{3}{4}$ by $\frac{4}{5}x+\frac{2}{3}$.

49. $\frac{2}{5}x-\frac{3}{4}y$ by $\frac{4}{3}x+\frac{5}{2}y$.

50. $2\frac{1}{2}x-3\frac{1}{3}$ by $1\frac{1}{5}x+\frac{3}{10}$.

51. $\frac{1}{2}a^2+\frac{1}{3}a+\frac{1}{4}$ by $\frac{1}{2}a-\frac{1}{3}$.

52. $\frac{9}{8}a^2+\frac{3}{2}a$ by $a-\frac{4}{3}$.

53. $\frac{3}{2}a^2-\frac{2}{3}ab+\frac{3}{4}b$ by $\frac{4}{3}a^2+\frac{3}{4}b$.

54. $\frac{1}{2}x^2-\frac{2}{5}x-\frac{3}{4}$ by $\frac{1}{2}x^2+\frac{2}{3}x-\frac{3}{4}$.

55. $\frac{7}{2}x^2-\frac{5}{3}xy+y^2$ by $\frac{2}{3}x+\frac{3}{4}y$.

56. $\frac{2}{3}m-\frac{5}{4}r+\frac{5}{2}p$ by $\frac{4}{5}m+\frac{3}{2}r-6p$.

57. $\frac{1}{2}a-\frac{2}{3}b+\frac{3}{4}c$ by $\frac{2}{3}a+\frac{3}{4}b-\frac{1}{2}c$.

EXERCISE 35.

EQUATIONS INVOLVING MULTIPLICATION.

Solve the following equations:

1. $9(23-5y)=8(5y-6)$.

Performing the multiplications by 9 and 8, we have

$$207-45y=40y-48.$$

Transposing the terms -48 and $-45y$, we get

$$207+48=40y+45y.$$

Uniting similar terms, $255=85y$;

whence, $y=3$.

2. $6(x-5)+2x=8x-2(x+10)$.

Performing the multiplications by 6 and 2, we have

$$(6x-30)+2x=8x-(2x+20).$$

Removing parentheses,

$$6x-30+2x=8x-2x-20.$$

Transposing the terms -30, $8x$, and $-2x$, we obtain

$$6x+2x-8x+2x=30-20.$$

Uniting similar terms, $2x=10$;

whence, $x=5$.

3. $2(x-1)=6$.
4. $4(x+5)=36$.
5. $7(y-3)=14$.
6. $19(x-7)=57$.
7. $13(12-z)=26$.
8. $17(13-x)=136$.
9. $5(35-x)=105$.
10. $7(135-x)=85$.
11. $8(2x+5)=152$.
12. $13(7z-61)=26$.
13. $4(15-2x)=20$.
14. $15(15-4x)=45$.
15. $3(x-5)+8=17$.
16. $3(x-3)+5=23$.
17. $15(x-3)-17=103$.
18. $17(17-x)+17=51$.
19. $6(37-x)+23=113$.
20. $5(2x+7)-8=57$.
21. $9(8x-5)+13=112$.
22. $5(x-1)=9x-25$.
23. $5x+(7-2x)=11$.
24. $8x-(3+5x)=9$.
25. $8(5-y)=3(y-5)$.
26. $10(x+2)=11x+17$.
27. $4(5x-3)=104-9x$.
28. $8(10-x)=5(x+3)$.
29. $7(15-3x)=6(x+4)$.
30. $8(9-2x)=5(3x+2)$.

31. $9(13-x)-4x=5(21-2x)+9x$.
32. $199+15x-(4x-5)=17(x+17)-13-22$.
33. $8(3x-2)-7x-5(12-3x)+28=8(3x+2)-32$.
34. $118x-13(5+11x)+15(3x-3)=18x-7(x-5)-119$
35. $7(3x-6)+5(x-3)+4(17-x)=44$.
36. $2(16-x)+3(5x-4)=12(3+x)-2(12-x)$.
37. $3(3x-2)-5(18-5x)=13(8x-12)+5(9x-11)$.

CHAPTER VI.

DIVISION.

EXERCISE 36.

DIVISION OF MONOMIALS BY MONOMIALS.

51. We may define **Division** as the process of *undoing* multiplication. In multiplication two factors are given to determine their product, while in division the product and one of the factors are given to determine the other factor. Thus, to divide 12 by 3, we must determine the factor which, when multiplied by the given factor 3, will produce 12.

In divison, the factor given is called the **Divisor**; the product given is called the **Dividend**; and the factor to be determined is called the **Quotient.**

1. What must $3a$ be multiplied by to produce $6a$? What, then, is $6a \div 3a$?

2. What must $3a$ be multiplied by to produce $6ab$? What, then, is $6ab \div 3a$?

3. What must $5b$ be multiplied by to produce $30abc^2$? What, then, is $30abc^2 \div 5b$?

4. What must a^3 be multiplied by to produce a^5? What, then, is $a^5 \div a^3$?

5. What must $3a^2x^3$ be multiplied by to produce $18a^5x^7$? What, then, is $18a^5x^7 \div 3a^2x^3$?

52. Since the dividend equals the product of the divisor and quotient, it follows that *the quotient of one monomial by another monomial is found by removing from the dividend all the factors which occur in the divisor.*

Thus $8abc \div 2ab = 4c$, *because* $2ab \times 4c = 8abc$, and the quotient $4c$ is found by removing the factors 2, a, and b from the dividend. If we wish to divide $2ab$ by xy, the factors x and y do not occur in the dividend, and consequently cannot be removed from it, so we can merely *indicate* the division thus: $\frac{2ab}{xy}$.

53. Since the product of two powers of the same number is found by adding the exponents, it follows that *the quotient of any power of a number divided by a lesser power of the same number, is equal to that number with an exponent equal to the exponent of the dividend minus the exponent of the divisor.* This is called the **Law of Exponents in Division.**

Thus, $a^5 \div a^2 = a^3$ *because* $a^3 \times a^2 = a^5$. Also $24a^3bx^5 \div 8abx^2 = 3a^2x^3$ because $8abx^2 \times 3a^2x^2 = 24a^3bx^4$.

54. The above principle may be stated in algebraic language by means of a *formula*, for if we let a stand for any number whatever, and n and r stand for *any whole* numbers whatever such that n is greater than r, we may say,

$$a^n \div a^r = a^{n-r},$$

because $$a^{n-r} \times a^r = a^n.$$

EXERCISE 37.

EXAMPLES.

Divide	Divide
1. $6a$ by 3.	6. $14cd$ by d.
2. $8b$ by 4.	7. $17mn$ by m.
3. $15mn$ by 5.	8. $28xy$ by x.
4. $49b^2$ by 7.	9. $25mx$ by $5x$.
5. $12ab$ by a.	10. $54a^2y$ by $6y$.

Divide

11. $18bx^5$ by $6b$.
12. $28cy^3$ by $4c$.
13. $108m^2c^5$ by $9m^2$.
14. $165bm^2$ by $15b$.
15. $76dx^3$ by $19d$.
16. $51m^2y$ by $17m^2$.
17. $63mn^5$ by $9m$.
18. $145mn^2y$ by $29m$.
19. $5cy$ by cy.
20. $48m^2n$ by m^2n.
21. $72x^5y$ by $72x^5$.
22. $17x^8y$ by $17x^8$.
23. $5bx^4$ by $5x^4$.
24. $71x^4y^2$ by $71y^2$.
25. $2m^3z$ by $2m^3z$.
26. $4y^5z^2$ by $2z^2$.
27. $17a^2b$ by a^2b.
28. $42bc^2$ by bc^2.
29. $2am^2$ by m.
30. $5a^4b^3$ by a^3.
31. $8m^3n^5$ by n^3.
32. $15ab^4$ by $3ab$.
33. $5b^5x^4$ by x^3.
34. $18x^2y^5$ by $3y^4$.
35. $3m^5n^2$ by n^3.
36. $9m^4y^5$ by $3m^2y^2$.
37. a^4b^5 by a^3b^2.
38. $15v^3x$ by $5v^2x$.
39. a^2bc^3 by ac^2.
40. d^5cx^4 by d^3x^2.

Divide

41. $7d^4x^5$ by d^3x^2.
42. a^7b^4 by ab.
43. m^5ny^2 by ny^2.
44. $21v^4x^2y^5$ by $7v^3x^2$.
45. $6x^7y^9$ by $3xy^8$.
46. $8c^5xy^4$ by $2c^2y^3$.
47. $51b^5x^4z$ by $17b^2xz$.
48. $5ax^7$ by $5ax^3$.
49. $15cd^4$ by $5d^3$.
50. $225m^5y^8$ by $15my^4$.
51. $121x^7y^8$ by $11x^5y^2$.
52. $1008b^4cd^5$ by $18b^3cd$.
53. $306m^4up$ by $18mp$.
54. $10a^2b$ by $5\ ab$.
55. $140x^3y^5$ by $35x^2y$.
56. $108h^2y^7$ by $9hy^6$.
57. $702s^5t^4$ by $18st$.
58. $1056x^6y^5$ by $96x^2y^3$.
59. $\frac{3}{5}x^2y^4$ by $\frac{5}{6}\ xy$.
60. $\frac{3}{7}m^6y^8$ by $\frac{2}{9}m^4y^5$.
61. $\frac{5}{7}a^7b^2$ by $\frac{5}{14}a^3b$.
62. $\frac{2}{11}p^2q$ by $\frac{7}{8}p$.
63. $\frac{1}{2}d^4c^5e$ by $\frac{7}{12}dc$.
64. $\frac{4}{5}p^4qr^5$ by $\frac{3}{7}p^3qr$.
65. $\frac{2}{7}\ kt^2$ by $\frac{5}{6}\ kt$.
66. $\frac{2}{3}c^4x^5$ by $\frac{7}{9}\ cx^4$.
67. $\frac{4}{5}a^2b^3$ by $\frac{8}{25}ab^2$.
68. $1\frac{1}{7}m^4x^3$ by $1\frac{4}{51}m^4x$.
69. $1\frac{7}{23}x^5y^4$ by $\frac{51}{46}xy^4$.
70. $.5gh^5$ by $.4h^4$.

EXERCISE 38.

DIVISION OF POLYNOMIALS BY MONOMIALS.

55. Since the product of a polynomial by a monomial is found by placing the monomial as a factor in each term of the polynomial, it follows that *a polynomial is divided by a monomial by removing from each term of the polynomial all the factors which occur in the divisor.*

Thus, $(6a-9b+15c)\div 3=2a-3b+5c$, *for* $(2a-3b+5c)\times 3=6a-9b+15c$, and the quotient is found by removing the factor 3 from each term of the dividend.

Also, $(9a^3x-15a^2x^2+12ax^3)\div 3ax=3a^2-5ax+4x^2$, because $(3a^2-5ax+4x^2)\times 3ax=9a^3x-15a^2x^2+12ax^3$, and the quotient is found by removing the factors 3, a, and x from each term of the dividend.

If we wish to divide $16a-9bx+3cx^2$ by $4x$, the factors 4 and x do not occur in some of the terms and consequently cannot be removed from them, so we can merely indicate the division in such cases, thus:

$$(16a-9bx+3cx^2)\div 4x=\frac{4a}{x}-\frac{9b}{4}+\frac{3cx}{4}.$$

56. The **Arrangement of the Work** when a polynomial is divided by a monomial is as follows:

Let it be required to divide $21m^2x^3-35am+7h^2m^2$ by $7m$.

Divisor, $7m$ | $21m^2x^3-35am+7h^2m^2$ Dividend,

$3m\ x^3-\ 5a\ +\ h^2m$ Quotient.

Let it be required to divide $16x^4+24x^3-20x^2$ by $4x^2$.

Divisor, $4x^2$ | $16x^4+24x^3-20x^2$ Dividend,

$4x^2+\ 6x\ -\ 5$ Quotient.

Let it be required to divide $5x^4-7x^3y+4x^2y^2$ by $5x^2$.

Divisor, $5x^2$ | $5x^4-7x^3y+4x^2y^2$ Dividend,

$x^2-\frac{7}{5}x\ y+\frac{4}{5}y^2$ Quotient.

EXERCISE 39.

EXAMPLES.

Divide

1. $15a^2-9a^5+18a^9$ by $3a^2$.
2. $8a^5b-24a^4b^3+16a^7b^8$ by $8a^4b$.
3. $6x^2y-4x^2z+6xyz$ by $2x$.
4. $5a^3b^3-35a^2b^2c^2+20abc^4$ by $5ab$.
5. $12a^2x^4y^2-24ax^3y^4-18x^2y+6xy$ by $6xy$.
6. $8x^4y^2-12x^2-16x$ by $4x$.
7. $3a^3b^2-6a^2b^4+12ab^6c$ by $3ab^2$.
8. $8x^2+16ax+6a^2x$ by $4x$.
9. $2a^2b^2-3ab^3+4a^3b-b^4$ by $3ab^3$.
10. $a^3x^3y-3a^2bx^2y+3ab^2xy^2-b^3xy^3$ by $abxy$.
11. $3x^4y+5x^3y-6x^2y^3-xy^4+4y^5$ by $2x^2y^2$.
12. $35x^3+15x^2y-xy^2-y^3$ by $5x^2$.
13. $6a^4b^3-42a^3b^4+9a^2b^5+16ab^6$ by $6a^2b^3$.
14. $5x^3y-25x^2y^2+10xy^3-8$ by $5xy$.
15. $4m^5r^2+\frac{2}{9}m^4r^5-\frac{6}{7}m^3r^6$ by $\frac{2}{3}m^3r$.
16. $36x^5y^4+18x^4y^5-16x^3y^6$ by $\frac{4}{5}x^3y^4$.
17. $.5m^2x^5-\frac{2}{9}m^4x^2+.35mx^6$ by $\frac{5}{6}mx^2$.
18. $\frac{3}{5}xy^5-.6x^2y-\frac{3}{5}\frac{3}{5}xy$ by $\frac{3}{5}xy$.

EXERCISE 40.

DIVISION OF POLYNOMIALS BY POLYNOMIALS.

57. Suppose we wish to divide x^2+5x+6 by $x+3$. The dividend x^2+5x+6 is the product of the divisor $x+3$ and another factor (the quotient), which we wish to find. Now, we can *undo* this multiplication if we can write x^2+5x+6 *so that* $(x+3)$ *will be a factor in each of*

its terms, for, by the previous exercise, an expression is divided by $(x+3)$ if we remove $(x+3)$ as a factor from each term. But we can say,

$$x^2+5x+6=x^2+3x+2x+6,$$

and, by using parentheses,

$$=x(x+3)+2(x+3).$$

Then, by removing the factor $(x+3)$ from each term, we obtain

$$(x^2+5x+6)\div(x+3)=x+2,$$

which is the required quotient.

As another example, let it be proposed to divide $x^2+14x+45$ by $x+9$. Now this can be done if we can write $x^2+14x+45$ so that $(x+9)$ will be a factor in each of its terms; for an expression is divided by $(x+9)$, if we remove $(x+9)$ as a factor from each term. But we know

$$x^2+14x+45=x^2+9x+5x+45$$

and, by using parentheses,

$$=x(x+9)+5(x+9).$$

Then, by removing the factor $(x+9)$ from each term, we obtain

$$(x^2+14x+45)\div(x+9)=x+5,$$

which is the required quotient.

Again, suppose it required to divide x^2+5x+4 by $x+4$. This can be done if we can write x^2+5x+x, so that $(x+4)$ is a factor in each of its terms; for an expression is divided by $(x+4)$, if we remove $(x+4)$ as a factor from each term. But we know

$$x^2+5x+4=x^2+4x+x+4$$

and by using parentheses,

$$=x(x+4)+(x+4).$$

Then, by removing the factor $(x+4)$ from each term, we obtain $(x^2+5x+4)\div(x+4)=x+1,$

which is the required quotient.

It is noticed in each of the above cases that our process consists in breaking the given dividend into parts, so that the divisor is a factor of each part, and then removing that factor from each term.

58. We may formally state the above process as follows:

One polynomial is divided by a secona polynomial, if the first polynomial be written so that the second polynomial is a factor in each term of the first, and this factor removed.

If it is impossible to write a polynomial in this manner, then that polynomial is not exactly divisible by the proposed divisor, and the division must be merely indicated. Thus, $(x^2+6x+2)\div(x+3)$ would be worked as follows:

$$x^2+6x+2=x^2+3x+3x+2.$$

Using parentheses $\quad =x(x+3)+(3x+2).$

Therefore,

$$(x^2+6x+2)\div(x+3)=x+\frac{3x+2}{x+3}.$$

EXERCISE 41.

EXAMPLES.

Divide	Divide
1. x^2+3x+2 by $x+2$.	9. $x^2+9x+14$ by $x+7$.
2. x^2+4x+3 by $x+3$.	10. $x^2+8x+15$ by $x+3$.
3. x^2+6x+5 by $x+5$.	11. $x^2+11x+28$ by $x+4$.
4. $x^2+7x+10$ by $x+2$.	12. $x^2+9x+18$ by $x+6$.
5. x^2+5x+6 by $x+3$.	13. $x^2+10x+24$ by $x+4$.
6. $x^2+9x+20$ by $x+4$.	14. $x^2+10x+21$ by $x+7$.
7. x^2+6x+8 by $x+2$.	15. $x^2+12x+35$ by $x+5$.
8. $x^2+8x+12$ by $x+6$.	16. x^2+4x+3 by $x+1$.

Divide

17. $x^2+4x-45$ by $x-5$:

$$x^2+4x-45=x^2-5x+9x-45.$$

Using parentheses $=x(x-5)+9(x-5).$

Removing the factor $(x-5)$ from each term

$$(x^2+4x-45)\div(x+5)=x+9,$$

the required quotient.

18. $x^2+3x-10$ by $x-2$.
19. x^2+3x-4 by $x-1$.
20. $x^2+3x-18$ by $x-3$.
21. $x^2-4x-21$ by $x-7$.
22. $x^2+2x-35$ by $x-5$.
23. $x^2-4x-12$ by $x-6$.

24. Divide $x^2-4x-45$ by $x+5$.

$$x^2-4x-45=x^2+5x-9x-45.$$

Using parentheses $=x(x+5)-9(x+5).$ Art. 23.

Removing the factor $(x+5)$ from each term

$$(x^2-4x-45)\div(x+5)=x-9,$$

the required quotient.

25. $x^2-3x-10$ by $x+2$.
26. x^2-3x-4 by $x+1$.
27. $x^2+3x-18$ by $x+6$.
28. $x^2-4x-21$ by $x+3$.
29. $x^2+3x-10$ by $x+5$.
30. $x^2+2x-35$ by $x+7$.

31. Divide $x^2-14x+45$ by $x-5$.

$$x^2-14x+45=x^2-5x-9x+45.$$

Using parentheses $=x(x-5)-9(x-5).$ Art. 23.

Removing the factor $(x-5)$ from each term

$$(x^2-14x+45)\div(x-5)=x-9$$

the required quotient.

32. $a^2-7a+10$ by $a-2$.
33. a^2-5a+4 by $a-1$.
34. $a^2-9a+18$ by $a-3$.
35. $a^2-10a+21$ by $a-7$.
36. $a^2-7a+10$ by $a-5$.
37. $a^2-10a+24$ by $a-6$.
38. $2a^2+10a+12$ by $a+3$.
39. $3a^2+19a+20$ by $a+5$.
40. $4a^2+14a+6$ by $4a+2$.
41. $4a^2+23a+15$ by $4a+3$.
42. $3a^2+10a+3$ by $a+3$.
43. $5a^2+11a+2$ by $a+2$.
44. $2a^2+11a+5$ by $2a+1$.
45. $3a^2+34a+11$ by $3a+1$.
46. $4a^2+23a+15$ by $4a+3$.
47. $24a^2-65a+21$ by $8a-3$.

EXERCISE 42.

ARRANGEMENT OF WORK IN THE DIVISION OF POLYNOMIALS BY POLYNOMIALS.

59. Since division is the process of undoing multiplication, we will exhibit in connection with each other the arrangement of work in the two operations.

MULTIPLICATION, OR THE DIRECT OPERATION.

Multiplicand,	$x + 4$
Multiplier,	$x + 9$
1st partial product,	$x^2 + 4x$
2d partial product,	$9x+36$
Product,	$x^2+13x+36$

DIVISION, OR THE INVERSE OPERATION.

	Divisor.	Dividend.	Quotient.
	$x+4$)	$x^2+13x+36$ (	$x+9$
1st partial dividend,		$x^2 + 4x$	
1st remainder,		$9x+36$	
2d partial dividend,		$9x+36$	
2d remainder,		0	

In the work in division the process is as follows: We first arrange dividend and divisor thus,

$$x+4\ )\ x^2+13x+36\ ($$

We next divide x^2, the first term of the dividend, by x, the first term of the divisor, which gives x as the quotient. We now multiply the *whole* divisor by x and put the product, x^2+4x, under the dividend. We then have

$$x+4\ )\ x^2+13x+36\ (\ x$$
$$x^2+\ 4x$$

by subtraction, $9x+36$

Then divide $9x$, the first term of this remainder, by x, the first term of the divisor, which gives 9 as the quotient. Now multiply the *whole* divisor by 9 and put the product, $9x+36$, under the last remainder. We then have

$$
\begin{array}{r}
x+4\ )\ x^2+13x+36\ (\ x+9 \\
\underline{x^2+\ 4x\ \ \ \ \ \ \ \ } \\
9x+36 \\
\underline{9x+36} \\
0
\end{array}
$$

by subtraction,

whence the quotient is $x+9$.

By comparing the above work in division with that of multiplication, the student will observe that the partial products which, when combined, constitute the final product in multiplication, occur in the work in division as "partial dividends," which, when combined, equal the original dividend. So that the process of division here used consists merely in breaking up the dividend into the component partial products, from each one of which *one* term of the quotient is obtained. Thus the above work is merely a convenient way of breaking $x^2+13x+36$ into the two expressions, ("partial dividends,")

$$(x^2+4x)+(9x+36),$$

which when written $x(x+4)+9(x+4)$

readily gives $x+9$ as the quotient by the method given in the last exercise.

60. We give a few more examples where multiplication and division are exhibited together, so that the student may more clearly understand this method of undoing multiplication.

(2)

$$
\begin{array}{r}
a\ +\ 7 \\
a\ +12 \\
\underline{a^2+\ 7a\ \ \ \ \ \ \ \ } \\
12a+84 \\
\underline{a^2+19a+84}\ (\ a+12 \\
a^2+\ 7a \\
\underline{12a+84} \\
12a+84
\end{array}
\qquad a+7\ )
$$

(3)

$$
\begin{array}{r}
2a-11 \\
3a+4 \\
\hline
6a^2-33a \\
\qquad 8a-44 \\
\hline
2a-11\ )\ 6a^2-25a-44\ (\ 3a+4 \\
6a^2-33a \\
\hline
\qquad 8a-44 \\
\qquad 8a-44 \\
\hline
\end{array}
$$

(4)

$$
\begin{array}{r}
a+9 \\
a-5 \\
\hline
a^2+9a \\
\qquad -5a-45 \\
\hline
a+9\ )\ a^2+4a-45\ (\ a-5 \\
a^2+9a \\
\hline
\qquad -5a-45 \\
\qquad -5a-45 \\
\hline
\end{array}
$$

Some prefer to write both divisor and quotient to the right of the dividend. Thus:

(5) Dividend, $15x^2-44x+32$ | $3x-4$ Divisor,

$15x^2-20x$ | $5x-8$ Quotient.

$-24x+32$

$-24x+32$

This arrangement saves space, and the divisor is where it is readily multiplied by each term of the quotient.

61. In examples (4) and (5) above the student will observe when the first term of the remainder is *subtractive* that the corresponding term found in the quotient has the *opposite* sign to the first term of the divisor. Of course the reason for this is, that the divisor must be multiplied by the term of the quotient to give the partial dividend,

and *it requires unlike signs to give minus in this multiplication.* If due attention be given to this fact no difficulty will be found in obtaining the correct sign for each term of the quotient.

In this statement it will be noticed that we have spoken of a term having no sign at all as if it had the sign +. See Art. 17.

62. *It is very important in the division of a polynomial by a polynomial that both dividend and divisor be arranged according to the powers of a common letter.* It makes no difference whether the arrangement be according to the descending or the ascending powers of a common letter, but both dividend and divisor should be arranged in the *same* order. Any letter may be selected for this purpose, but the letter which occurs the greatest number of times in the given dividend and divisor, is naturally preferred.

If some powers of the selected letter do not occur in the dividend, then it is well to leave a blank space in the work for every such term. Thus:

(1) Divide a^2-b^2 by $a+b$.

$$
\begin{array}{l}
a+b \;)\; a^2 \quad\quad -b^2 \;(\; a-b \\
\quad a^2+ab \\
\quad\quad -ab-b^2 \\
\quad\quad -ab-b^2
\end{array}
$$

(2) Also, divide a^3-b^3 by $a-b$.

$$
\begin{array}{l}
a-b \;)\; a^3 \quad\quad\quad\quad -b^3 \;(\; a^2+ab+b^2 \\
\quad a^3-a^2b \\
\quad\quad a^2b \\
\quad\quad a^2b-ab^2 \\
\quad\quad\quad\quad ab^2-b^3 \\
\quad\quad\quad\quad ab^2-b^3
\end{array}
$$

EXERCISE 43.

EXAMPLES.

Divide

1. $x^2+3x-40$ by $x-5$.
2. x^2+5x-6 by $x-1$.
3. $x^2+7x-30$ by $x-3$.
4. x^2+4x-5 by $x-1$.
5. $x^2-2x-63$ by $x-9$.
6. $x^2+7x-44$ by $x-4$.
7. x^2-4 by $x-2$.

Divide

8. $a^2+4a-45$ by $a-5$.
9. $a^2-4a-32$ by $a-8$.
10. $a^2+7a-78$ by $a-6$.
11. a^2-121 by $a+11$.
12. $x^2-ax-6a^2$ by $x-3a$.
13. x^2-9a^2 by $x-3a$.
14. x^2-49y^2 by $x+7y$.
15. $a^2+6ab+9b^2$ by $a+3b$.
16. $a^2-17ab+72b^2$ by $a-9b$.
17. $2x^2-9x+10$ by $2x-5$.
18. $3x^2+2x-1$ by $3x-1$.
19. $6x^2+5x+21$ by $3x+7$.
20. $9x^2-64$ by $3x+8$.
21. $9x^2-3xy-2y^2$ by $3x-2y$.
22. $4x^2+4xy-35y^2$ by $2x+7y$.
23. $2x^2+5ax-25a^2$ by $x+5a$.
24. $4x^2+4ax+a^2$ by $2x+a$.
25. $5a^6+15a^5+5a+15$ by $a+3$.

$$
\begin{array}{l}
a+3\ )\ 5a^6+15a^5+5a+15\ (\ 5a^5+5 \\
\quad\underline{5a^6+15a^5} \\
\qquad\qquad 5a+15 \\
\qquad\qquad \underline{5a+15}
\end{array}
$$

26. $42a^4+41a^3-9a^2-9a-1$ by $7a^2+8a+1$.

$$
\begin{array}{l}
7a^2+8a+1\ )\ 42a^4+41a^3-\ 9a^2-9a-1\ (\ 6a^2-a-1 \\
\quad\underline{42a^4+48a^3+\ 6a^2} \\
\qquad -\ 7a^3-15a^2-9a \\
\qquad\underline{-\ 7a^3-\ 8a^2-\ a} \\
\qquad\qquad -\ 7a^2-8a-1 \\
\qquad\qquad \underline{-\ 7a^2-8a-1}
\end{array}
$$

27. Divide $27a^3+8b^3$ by $3a+2b$.

$$
\begin{array}{l|l}
3a+2b & 27a^3 \qquad\qquad +8b^3 \;(\; 9a^2-6ab+4b^2 \\
\hline
 & \underline{27a^3+18a^2b} \\
 & \quad -18a^2b \\
 & \quad \underline{-18a^2b-12ab^2} \\
 & \qquad\qquad 12ab^2+8b^3 \\
 & \qquad\qquad \underline{12ab^2+8b^3}
\end{array}
$$

28. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

Arrange according to the descending powers of one of the letters, say a. It is important to *keep* this arrangement throughout the work.

$$
\begin{array}{l|l}
a+b+c & a^3-3abc+b^3+c^3 \;(\; a^2-ab-ac+b^2-bc+c^2 \\
\hline
 & \underline{a^3+a^2b+a^2c} \\
 & \quad -a^2b-a^2c \qquad -3abc \\
 & \quad \underline{-a^2b \qquad -ab^2- \;abc} \\
 & \qquad -a^2c+ab^2-2abc \\
 & \qquad \underline{-a^2c \qquad -\;abc-ac^2} \\
 & \qquad\qquad ab^2- \;abc+ac^2+b^3 \\
 & \qquad\qquad \underline{ab^2 \qquad\qquad +b^3+b^2c} \\
 & \qquad\qquad\quad -\;abc+ac^2 \qquad -b^2c \\
 & \qquad\qquad\quad \underline{-\;abc \qquad\qquad -b^2c-bc^2} \\
 & \qquad\qquad\qquad\qquad ac^2 \qquad\qquad +bc^2+c^3 \\
 & \qquad\qquad\qquad\qquad \underline{ac^2 \qquad\qquad +bc^2+c^3}
\end{array}
$$

Divide

29. $2a^4-6a^3+3a^2-3a+1$ by a^2-3a+1.

30. $2m^4-6m^3+3m^2-3m+1$ by m^2-3m+1.

31. $6a^3x-17a^2x^2+14ax^3-3x^4$ by $2a-3x$.

32. $4y^4-18y^3+22y^2-7y+5$ by $2y-5$.

33. $4a^3+4a^2-29a+21$ by $2a-3$.

34. $45x^4+18x^3+35x^2+4x-4$ by $9x^3+7x-2$.

35. $\frac{9}{10}a^4-\frac{4}{5}a^3b+\frac{19}{36}a^2b^2+\frac{1}{6}ab^3$ by $\frac{3}{2}a+\frac{1}{3}b$.

36. $a^4b^2-b^6+a^6-a^2b^4$ by $a^3+a^2b+b^3+ab^2$.

37. $x^3+y^3+3xy-1$ by $x+y-1$.

38. $a^2-2ab+b^2-c^2+2cd-d^2$ by $a-b+c-d$.

Divide

39. $a^4+b^4-c^4-2a^2b^2$ by $a^2-b^2-c^2$.

40. $1+x^8+x^4$ by x^2+1-x.

41. a^5-243 by $a-3$.

42. $1-6x^5+5x^6$ by $1-2x+x^2$.

43. $x^6-2a^3x^3+a^6$ by $x^2-2ax+a^2$.

44. $3x^3-3$ by $\frac{1}{2}x^2+\frac{1}{2}x+\frac{1}{2}$.

CHAPTER VII.

NEGATIVE QUANTITIES.

EXERCISE 44.

NUMBER AND QUANTITY.

63. Anything which can be measured by a unit of the same kind is called a **Quantity**. Thus, 10 bushels is a quantity, the unit being a bushel, and this unit taken 10 times gives the *quantity*, 10 bushels. Also 10 cords is a quantity, the unit in this case being one cord, and this unit taken 10 times gives the *quantity*, 10 cords.

Also the abstract number 10 is a quantity, the unit in this case being the abstract number 1, and this unit taken 10 times gives the quantity 10. Of course, the unit itself is a quantity.

The word quantity as above defined, plainly includes number, but while a number is a quantity, a quantity is not always a number.

Five miles would be called a *quantity* and *never* be called a *number*, but the number 5 may be called either a number or a quantity indifferently.

The word quantity is usually used as here explained, but some writers on Algebra never use the word quantity to include number.

64. The answer to a problem in Algebra is often something like 5 *miles* or 4 *tons* or 3 *dollars*, or some other *concrete* quantity, but the reasoning is always conducted by *numbers*, and so the letters used in Algebra *always represent numbers*, and the result reached is the

number of miles or tons or dollars, or whatever it may be, and then the name of the thing we are considering may be added at the end to the *number* we have obtained by working the problem.

EXERCISE 45.

OPPOSITE DIRECTIONS.

1. If a man travel east 30 miles and then west 20 miles, how far will he be from the starting point?

2. If he travel east 30 miles and then west 40 miles, how far will he be from the starting point?

3. If the temperature is zero, and it rises 10 degrees and then falls 6 degrees, how far will it then be from zero?

4. If it rises 10 degrees and then falls 14 degrees, how far will it then be from zero?

5. If a man receive 50 dollars and spend 35 dollars, the amount of money he has, differs from what he had before by how much?

6. If he receive 50 dollars and spend 65 dollars, the amount of money he has, differs from what he had before by how much?

7. If a man travel east 30 miles and then west 20 miles, is he east or west of the starting point, and how far?

8. If he travel east 30 miles and then west 40 miles, is he east or west of the starting point, and how far?

9. If the temperature is zero and it rises 10 degrees and then falls 6 degrees, will it then be above or below zero, and how far?

10. If the temperature is zero and it rises 10 degrees and then falls 14 degrees, will it be above or below zero, and how far?

11. If a man receive 50 dollars and spend 35 dollars, has he more or less than he had before, and how much?

12. If he receive 50 dollars and spend 65 dollars, has he more or less than he had before, and how much?

65. It is plain that the last six of these examples are very much like the first six, and yet these two sets differ in one important respect. In the first six we are concerned only with amount, in the last six something besides amount is required, and this something we may, for want of a better word, call **Direction.**

Illustrations might have been given involving other kinds of quantity; as for example, degrees *north* or *south* of the Equator, longitude *east* or *west,* time *before* or *after* a given event, *gain* or *loss*, etc.

Plainly in those cases where we consider direction, there are two directions, either one of which is just the reverse, or opposite, of the other one; but where we are concerned only with amount, there is no such thing as reverse.

Time cannot be reversed. But time *before* can be reversed, and the reverse of it is time *after.*

Distance cannot be reversed. But distance *east* can be reversed, and the reverse of it is distance *west*, etc.

EXERCISE 46.

How Directions are Distinguished.

66. Sometimes in Algebra we are not called upon to consider anything but amount, or magnitude, and sometimes we are obliged to consider, both magnitude and direction, and so we must in some way distinguish between the two opposite directions. To find out how to distinguish between these opposite directions, let us consider the following examples.

The distinction between positive and negative is made by means of the signs + and −. To explain this let us consider two or three questions:

1. If a man has $1000, and he gains $500, and then looses $700, how much will he then have? How much less is $1000+500-700$ than 1000?

2. A man had a dollars and gained 500 dollars, and then lost 700 dollars, how much did he then have? How much less is $a+500-700$ than a? Write an expression of two terms which shall be equal to $a+500-700$.

The two terms $+500-700$ can be replaced by what single term?

3. A man, 100 miles east of St. Louis, travels east 50 miles and then west 75 miles, how far east of St. Louis is he then? A man, a miles east of St. Louis, travels 50 miles east and 75 miles west, how far is he then east of St. Louis? Is his distance east expressed by $a+50-75$? Is it also expressed by $a-25$?

67. Notice in these examples that two terms, one additive and one subtractive, may be replaced by a single term which in each example given has been a subtractive term. Notice also, that in the expressions with three terms, the additive and subtractive terms express opposite directions, as in the last example the + sign was used in connection with the distance traveled east, and the − sign used in connection with the distance traveled west.

We are naturally led to *associate the + sign with one direction and the − sign with the other direction.*

In example 3, we could replace $+50-75$ by -25. Now we know that a journey of 50 miles east, followed by one of 75 miles west, is equivalent to a single journey

of 25 miles west, so the term -25 which could be substituted for $+50-75$ comes to have a meaning, namely 25 miles *west* in this example.

In example 2, we could replace the two terms $+500$ -700 by the single term -200, and here, also, a gain of \$500 followed by a loss of \$700, is the same as a single loss of \$200, so that the term -200, which could be substituted for $+500-700$, comes to have a meaning, namely, \$200 *loss* in this example.

Many other illustrations might be given, but these are enough to show that *we distinguish between the two opposite directions by means of the signs* $+$ *and* $-$.

We further see that when the distinction is so made, *each term of the expression comes to have a meaning*, and so *it does not matter which term is written first.*

68. We see from this that **the Signs Plus and Minus have a Double Use** in Algebra; first, to indicate the operations of addition and subtraction respectively; second, to distinguish between opposite directions as just described. There is no danger of confusion arising from this double use; the context will always make it clear in which sense the signs are used.

EXERCISE 47.

POSITIVE AND NEGATIVE NUMBERS.

69. In Arithmetic we are concerned only with the numbers 0, 1, 2, 3, 4, etc.,
and intermediate numbers; but in Algebra we consider *besides* these the numbers
0, -1, -2, -3, -4, etc.,
and intermediate numbers.

We may represent the two classes of numbers considered in Algebra on the following scale:

. . $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5,$ . .

which extends indefinitely in both directions from zero.

The sign + perhaps ought to precede each of the numbers at the right of zero on this scale, but we will agree that when no sign is written before a number the sign + is understood.

70. Numbers to the *right* of zero in the above scale are called **Positive,** and those to the *left* of zero are called **Negative,** or, we might say, numbers represented by figures preceded by a + sign or no sign at all are *positive* and numbers represented by figures preceded by a − sign are *negative.*

71. In Algebra numbers are often represented by letters, and we have already seen that a letter may stand for a whole number or a fraction. We will now further extend the signification of a letter by allowing that it may stand for one of the numbers to the left of zero in the above scale as well as one to the right of zero; so that while in the case of a number represented by figures we can tell whether the number is positive or negative by the sign that precedes it, yet in the case of a number represented by a letter, we cannot tell by the sign before it whether it is positive or negative.

A minus sign before a number always represents a number of the opposite kind from that represented by the same number with a plus sign or no sign at all before it.

We know that the number 5 is *positive,* but we do not know whether a is positive or negative until we know its value.

We know that -5 is *negative*, but we do not know whether $-a$ is positive or negative until we know the value of a.

If $a=3$, then $-a=-3$, and a is positive and $-a$ is negative; but if $a=-3$, then $-a=3$ (because 3 is the opposite of -3), and a is negative and $-a$ is positive.

72. We have seen that we are required to distinguish between quantities opposite to each other and that this distinction is made by means of the signs plus and minus; for example, if $+10$ degrees means a temperature of 10 degrees above zero, then -10 degrees would mean a temperature of 10 degrees below zero, and if $+10$ miles means 10 miles north of the equator, then -10 miles would mean 10 miles south of the equator, and if $+10$ rods means 10 rods east of a given point, then -10 rods would mean 10 rods west of the same given point, and if $+10$ be 10 units of *any* kind in *any* sense, then -10 would be 10 units of the *same* kind in just the *opposite* sense.

In each case one of these quantities is positive and the opposite one is negative. Either direction may be selected as positive, and then, of course, the opposite direction will be negative. But it is almost always easiest and best to select the quantity about which we are inquiring in any given problem as positive.

Anything that *increases* the quantity we have chosen as positive will also be positive, and anything that *decreases* our selected positive quantity will be negative.

EXERCISE 48.

ILLUSTRATIVE EXAMPLES.

1. One day a man travels east 100 miles, the next day he travels west 250 miles, and the third day he travels east 175 miles; how far is he then east of his starting point?

SOLUTION.

Because the problem asks how far *east* the man is of his starting point, we take distance *east* to be positive, and therefore, distance west to be negative. We associate distance east with the + sign, and distance west with the − sign. The distance the man is east of his starting point will, therefore, be represented by

$$100-250+175$$

which is equal to 25. Hence, the man is 25 miles east of his starting point.

2. One day a man travels east 100 miles, next day he travels west 250 miles, and the third day he travels east 50 miles; how far east is he then from his starting point?

SOLUTION.

Because the problem asks how far east the man is of his starting point, we take as before distance *east* to be positive, and of course, distance west to be negative, and asssociate distance east with the + sign and distance west with the − sign.

The distance the man is east of his starting point will then be represented by $100-250+50$.

Now, these three terms may be replaced by -100, hence we say $100-250+50=-100$. Therefore, the distance from the starting point is represented by -100, which by our interpretation of negative quantities means 100 miles *west* of the starting point.

SECOND SOLUTION.

Let us now take distance *west* to be positive, and therefore distance east negative, and everywhere associate distance west with the + sign, and distance east with the − sign. Upon this assumption the distance the man is west of his starting point will be represented by

$$-100+250-50$$

which is equal to 100. Hence, as before, the man is 100 miles *west* of his starting point.

3. A boy rows his boat up stream 3 miles in an hour, and then rests an hour, when he floats down stream 5 miles, he then rows up stream again 3 miles in the next hour; how far is he above the starting point?

Between what two directions are we required to distinguish in this problem? Which is it most natural to take for the positive direction, up stream or down stream?

4. A boy rows his boat up stream 3 miles in an hour and then rests an hour, when he floats down stream 6 miles, he then rows up stream again 2 miles in the next hour; how far is he above his starting point?

5. A boy rows his boat up stream 3 miles in an hour and then rests an hour, when he floats down stream 6 miles, he then rows up stream again 2 miles in the next hour; how far is he then below his starting point?

6. A merchant was in business 3 years; the first year he lost $2000, the second year he gained $500, and the third year he gained $2600. What was his profit in business?

7. A merchant was in business 3 years; the first year he lost $3000, the second year he gained $500, and the third year he gained $2000. What was his profit in business?

8. A merchant was in business 3 years; the first year he lost $3000, the second year he lost $500, and the third year he gained $2500. What was his profit in business?

9. A merchant was in business 3 years; the first year he lost $3000, the second year he lost $500, and the third year he gained $2500. What was his loss in business?

10. A merchant was in business 4 years; the first year he lost $2550, the second year he gained $1025, the third year he gained $575, and the fourth year he gained $900. Did he gain or lose, and how much?

11. On a cold morning the temperature was 10 degrees below zero, and it rose 12 degrees during the day; what was the temperature at evening?

12. On a cold morning the temperature was 10 degrees below zero, and it rose 12 degrees during the day, and fell 6 degrees during the night; what was the temperature the next morning?

73. FUNDAMENTAL OPERATIONS WITH NEGATIVE NUMBERS. Until the beginning of the present chapter the letters used always stood for positive numbers, but now a letter may stand for either a positive or a negative number. Moreover, a letter standing alone with a minus sign before it has a meaning now, but to know the meaning of a letter with a minus sign before it, as, for instance, $-a$, we must first know what a means when preceded by a $+$ sign or by no sign at all, and then give to $-a$ the opposite interpretation. As letters now have a much larger significance than before, we must briefly re-examine the four fundamental operations of Algebra, viz.: Addition, Subtraction, Multiplication, and Division, to see if the results previously obtained, on the supposition that the letters stood for positive numbers, hold when the letters stand for negative as well as for positive numbers.

EXERCISE 49.

ADDITION.

74. Addition is the process of finding the result of two or more numbers taken together. The result found is the sum.

This definition agrees with all before found, when all the numbers used are positive, and, as we shall see, is sufficiently broad to include the case of negative numbers.

If we wish to add $+25$ and -15 together, we get for our result $+25-15$ or $+10$.

This is called adding -15 to $+25$, and is easily seen to be the same as subtracting $+15$ from $+25$.

With this definition of addition, it is easy to see that to find the sum of two or more numbers, we supply $+$ signs to terms having no signs, and then, write them down with signs unchanged, and combine terms if possible, the same as in Chapter III.

To add 7, 5, and -8 we supply $+$ signs before the terms 7 and 5, and write

$$+7+5-8.$$

When the sign of the first term is $+$ we may omit the sign if we like, but we must never omit the $-$ sign, and never the $+$ sign in any term but the first.

Addition is nicely illustrated by the scale of algebraic numbers. For example, if we wish to add 6 to -4 we begin at -4 on the scale and count forward 6 spaces, arriving thereby at 2. Again, to add -6 to -4, we begin at -4 on the scale and count backward 6 spaces, arriving thereby at -10.

Now, if we use letters, we can easily find the sum by the method here given.

Suppose we wish to add $a, -b,$ and $-c$. Supplying $+$ sign before a, and writing down with signs unchanged we get, $+a-b-c$.

Suppose next we wish to add $a+b$, $b-c$, and $e-f$.

Supplying $+$ signs before terms a, b, and e, and writing down with signs unchanged we get,

$$+a+b+b-c+e-f$$

or

$$a+2b-c+e-f.$$

Exactly the result obtained by the method of Chapter III.

Any result reached in Chapter III could easily be tested by the notion of addition here presented, and found correct. Therefore, the methods and results of Chapter III hold, whether the numbers used are positive or negative.

One important difference may now be noticed between addition in Arithmetic and Algebra. In Arithmetic addition implies augmentation, but in Algebra this is not necessarily the case.

EXAMPLES.

1. Add $a+b+c$, $-3a-b+c$, and $-a-b-4c$.

2. Add $-r-2s-3t$, $-r-t$, and $-2r-2s$.

3. Add $a-b+c$, $-a+b+c$, and $a+b-c$.

4. Add $a+b$, $-\frac{1}{2}a$, $-\frac{2}{3}b$, and $4c$.

5. If $x=a+b-2c$, $y=a-2b+c$, and $z=-2a+b+c$ show that $x+y+z=0$.

6. If $x=a+b-c$, $y=a-b+c$, and $z=-a+b+c$ show that $x+y+z=a+b+c$.

EXERCISE 50.

SUBTRACTION.

75. Subtraction is the reverse of addition *i. e.*, it is the process of finding a number called the remainder, which added to the subtrahend will give the minuend. For example, 12 is a number, which added to -5 will give 7; therefore, -5 subtracted from 7 gives 12.

It is easy to see from this meaning of subtraction that to subtract one number from another, we *change the sign of the subtrahend and then unite it to the minuend.*

This is exactly the method of Chapter IV; therefore, the methods and results of Chapter IV hold, when the letters used stand for negative as well as positive numbers.

EXAMPLES.

1. From x^2+y^2 take x^2-y^2.

2. From $x+a-3b^4$ take $-x-3a+b^4$.

3. From $a-b+c-d$ take $a+b-c+d$.

4. From $2n^3+3a^3-r^3-s^3$ take $n^3-a^3+r^3-2s^3$.

5. From $a^2+2ab+b^2$ take $a^2-2ab+b^2$.

6. From $uvw^2-2uv^2w+3u^2vw$ take $3uvw^2+2uv^2w+u^2vw$.

7. From the sum of a^2+b^2 and $-2ab$ subtract the sum of a^2-b^2 and $3c^3$.

8. From $x^3+ax^2+a^2x+a^3$ subtract $2ax^2-a^2x$, and from this difference subtract $2ax^2-a^2x$.

9. What must be added to $r^2+s^2+t^2$ to produce 3?

10. What must be subtracted from abc^3 to produce $m+r$?

11. What must $9ab$ be subtracted from to produce $-ab$?

EXERCISE 51.

Multiplication.

76. The definition of multiplication given in Chapter V is sufficient to include the case of negative numbers.

What is the product of 6 multiplied by -3? To produce -3 from unity we take unity three times and reverse the result. Hence to multiply 6 by -3 we must take 6 three times and reverse the result, giving -18.

What is the product of -6 multiplied by 3? To produce 3 from unity we take unity 3 times; hence to multiply -6 by 3 we take -6 three times, giving -18.

What is the product of -6 multiplied by -3? To produce -3 from unity we take unity 3 times and then reverse the result; hence to multiply -6 by -3 we take -6 three times, giving -18, and reverse the result, giving $+18$.

From these illustrations it is evident that numbers can be multiplied as in Chapter V whether the letters used stand for positive or negative numbers. It is also evident that we have here another and very nice demonstration of the "*Law of Signs*" in multiplication.

Examples.

Find the product of

1. 10 multiplied by -4.
2. -25 multiplied by 4.
3. $-\frac{7}{2}$ multiplied by $-\frac{2}{14}$.
4. abc multiplied by $-a^3b^2c$.

5. $-\frac{2}{3}ms^3$ multiplied by $\frac{3}{2}m^3s$.
6. a^3+b^3 multiplied by $-a^3+b^3$.
7. $-5abc-3rs$ multiplied by $5abc-3rs$.
8. $-a-b-c$ multiplied by $-a+3b-9c$.
9. $5r^3+6s^3$ multiplied by $-7r+2s-t$.
10. $-x^5y^4-xz^3+yz^5$ multiplied by $4xy-\frac{7}{8}y^3$.

EXERCISE 52.

DIVISION.

77. Division is the reverse of multiplication, that is, it is the process of finding a number called the quotient, which multiplied by the divisor, equals the dividend. We may express this in the form of an equation thus,

Divisor × *Quotient* = *Dividend.*

As here arranged this is a case of multiplication where the divisor is the multiplicand, the quotient is the multiplier, and the dividend is the product. As we know in multiplication, that when the multiplier has the sign + the signs of the multiplicand and product are alike, it follows here that when the quotient has the sign + the dividend and divisor must have like signs, or stated in the reverse order, *when the signs of the dividend and divisor are alike the sign of the quotient is* +.

It is also easy to see that when the quotient has the sign − the signs of the dividend and divisor are unlike, or stated in reverse order. *When the signs of the dividend and divisor are unlike, the sign of the quotient is* −.

These two statements put together give the **Law of Signs in Division,** viz.: *In division like signs give* + *and unlike signs give* −.

EXAMPLES.

1. Divide 10 by -5.
2. Divide 5 by -10.
3. Divide 27 by -3.
4. Divide -63 by 7.
5. Divide -63 by -7.
6. Divide a^2 by $-a$.
7. Divide a^5b^3 by $-a^4$.
8. Divide $-a^5b^5c^5$ by a^4b^2c
9. Divide $m^3r^5s^7$ by $-ms^4$.
10. Divide $6a^3-a^2-14a+3$ by $3a^2+4a-1$.

CHAPTER VIII.

PARENTHESES.

EXERCISE 53.

REMOVAL OF PARENTHESES.

78. The subject of parentheses has already been considered to some extent, and we have already learned that an expression within a parenthesis is to be looked upon as a single number just as though it were represented by a single symbol.

Now, it may happen that an expression within a parenthesis is itself an expression which contains a parenthesis, so we would have a parenthesis within a parenthesis. Indeed, we may have several parentheses one within another.

These complicated expressions present no difficulty, for we can take the parentheses one at a time, and if we know how to remove one, we may do this and then remove another, and so on until all are removed. For example, if we wish to remove the parentheses from

$$a+(b-(c-d)+e),$$

we begin by removing the inner parenthesis first and write the expression in the form

$$a+(b-c+d+e),$$

and now by removing the remaining parenthesis we write the result in the final form

$$a+b-c+d+e.$$

79. In removing parentheses it is usually best to remove the innermost parenthesis first, and then the innermost parenthesis of all that remains, and so on until all, or as many as may be desired, are removed.

80. When several parentheses are used one within another, they are often made of different shapes and sometimes of different sizes to prevent confusion. Some of the forms used are, ( ), { }, [ ]. Sometimes a horizontal line, called a **Vinculum** or bar, is drawn above an expression instead of using a parenthesis. Thus,

$$a+\overline{x-y}$$

means the same as $a+(x-y)$.

Remove the parentheses from the following expressions:

1. $a+[b-(c+d)]$.
2. $a+[b-(c+d-e)+2]$.
3. $3a+2b-[(5b^2-4)-(4b-2)]$.
4. $m^2+a^2-(r^3-[b^3+5(2x+1)]-c)$.
5. $4x+3y-5x-[2y-(6x-6y)]$.
6. $4a-(6a-[5a-(4a-2a)])$.
7. $3x-(4y-[5x-(6y-7x)])$.
8. $3x+(-4y+[5x-(6y-7x)])$.
9. $5a^3-(4b^3-[3(a^2+b^2)-4(x-2)])+2$.
10. $1-\{1-(1-[1-(1-x)])\}+x$.

EXERCISE 54.

INSERTION OF PARENTHESES.

1. If the parenthesis be removed from $a+(b-c+d)$, what is the result?

2. If the parenthesis be removed from $a-(-b+c-d)$ what is the result?

Notice how the expressions within the parentheses in these two examples are related, and notice, also, how the results are related.

3. If from any expression a parenthesis preceded by a + sign be removed, what is the effect on the terms originally within the parenthesis?

4. How, therefore, can any number of terms be brought within a parenthesis, preceded by a + sign?

5. If from any expression a parenthesis preceded by a − sign be removed, what is the effect on the terms originally within the parenthesis?

6. How, therefore, can any number of terms be brought within a parenthesis, preceded by a − sign?

7. Enclose the last three terms of $a^2+b^3-c^4+5$ in a parenthesis, preceded by a + sign.

8. Enclose the last three terms of $a^2+b^3-c^4+5$ within a parenthesis, preceded by a − sign.

9. Enclose the last two terms of $a^2-b^3-c^4+5$ within a parenthesis, preceded by a + sign.

10. Enclose the last two terms of $a^2+b^3-c^4+5$ within a parenthesis, preceded by a − sign.

11. Enclose the last three terms of $a+b-4c-5e^2+6r^3-n^4-16$ within a parenthesis, preceded by a + sign.

12. Enclose all but the first term of $a+b-4c+5e^2+6r^3-n^4-16$ within a parenthesis, preceded by a − sign.

13. Enclose the third and fourth terms of $a+b-4c+5e^2+6r^3-n^4-16$ within a parenthesis, preceded by a − sign, and the fifth, sixth, and seventh terms of the same expression in another parenthesis, preceded by a − sign.

14. Fill out the blank parenthesis in the equation $a^2+(2b-1)=a^2-(\quad)$.

15. Fill out the blank parenthesis in the equation $a^5b-[5cd-3x^2-(7t^4-9)]=a^5b+(\quad)+7t^4-9$.

CHAPTER IX.

ELEMENTARY FACTORS, MULTIPLES, AND FRACTIONS.

EXERCISE 55.

FACTORS.

81. A definition of factor has been given, (see page 12), and we have already learned that an expression may have several factors. For example, the different factors of $10x^2$ are

$$2,\ 5,\ 10,\ x,\ 2x,\ 5x,\ 10x,\ x^2,\ 2x^2,\ 5x^2.$$

Of these factors 2, 5 and x may be called **Prime**, because they cannot be further factored.

The expression $10x^2$ contains the prime factor x *twice*, so *all* the prime factors of $10x^2$ are 2, 5, x, x; and as any expression equals the product of all its prime factors, we have

$$10x^2 = 2 \times 5xx.$$

When an expression is written as the product of all its prime factors, it is said to be **Resolved into its Prime Factors.**

Resolve the following eight expressions into their prime factors:

1. $30x^4y^3$. 3. $38ab^2c^3$. 5. $133r^2s^2$. 7. $2431uv^2w^3$.
2. $150a^5b^2$. 4. $51m^2r^3$. 6. $429abn$. 8. $259x^3yz^4$.
9. Find three of the factors $30x^4y^3$ which are not prime.
10. What are the *different* prime factors of $15a^3$?
11. What are *all* the prime factors of $15a^3$?
12. Resolve $8a^3b^3c^3$ into its prime factors.

EXERCISE 56.

HIGHEST COMMON FACTOR.

1. Does the factor a belong to each of the two expressions $5a^2x$ and $7ax^2$? Does the factor x belong to each of these two expressions? Does the factor ax belong to each expression?

When the same factor belongs to two or more expressions it is called a **Common Factor** of those expressions.

When a *prime* factor is common to two or more expressions it is called a **Common Prime Factor** of those expressions.

2. If 2 and 3 are each factors of an expression, is 6 a factor of that expression?

3. If a and b are each factors of an expression, is ab a factor of that expression?

4. If a and b are each *common* factors of two or more expressions, is ab a common factor of those expressions?

5. Is the product of any number of common factors of two or more expressions, a common factor of those expressions?

82. The product of *all* the common prime factors of two or more expressions is called the **Highest Common Factor** of those expressions. The abbreviation H. C. F. is frequently used to stand for the highest common factor.

83. From the definition of the H. C. F. it follows at once that the way to find H. C. F. of two or more expressions, is to *resolve each expression into its prime factors, and take the product of all those which are common to all the expressions.*

Find the H. C. F. of the following expressions :

1. x^3y and x^2y^2.
2. $8abc^2$ and $12a^4b$.
3. $6rs^2$ and $15r^2st^3$.
4. $7a^2bc^3$ and $14a^3c^5$.
5. $7uvw$ and $10v^2wx^4$.
6. x^3y, $8xyz$ and $10x^2z^2$.
7. $3x^2yz$, $15xzw^3$ and $12xyw$.
8. $4mn^3$, $8mn^5x$, and $12xmn^6$
9. $50a^3b^3c^3$, $75a^4bcd$, and $80\,a^2bd$.
10. $5rs^4t^6$, $7r^6sx^4$, $9r^2x^3$, and $11rstx$.
11. $35x^3y$, $63ax^2z^3$, and $70abx^4yz$.
12. $17a^5b^6c^7d^8$, $51m^3a^4b^3c^5d^4$, and $34ra^3b^4c^3d^3$.
13. x^2yz, xy^2z, xyz^2, and $x^2y^2z^2$.
14. $2rstuv$, $3r^2stu$, $4rs^3tv$, and $5r^2s^3tw^3$.
15. $7u^4v^6w^7$, $35mnu^2v^3$, $136r^3u^6v^8w$, and $77uv^4w$.
16. $21a^3b^4c^6d^8$, $63a^9b^7c^6xyz$, $84a^2b^3c^5uvw$, and $49a^6b^2cx$.
17. $99x^2$, $187x^3y$, $253yz$, $143uv$, and $275x^4$.
18. $2a^2n^3x^4$, $12bn^5y$, $100c^5n^4x$, and $4en^6xy^9$.
19. $16a^5bx^9$, $80b^2ny^4$, $36abr^3x^7$, and $28b^5cdy^4$.
20. $13x^3y^4z^6$, $90x^3yz$, $54xy^6$, $360y^4z$, and $36xz$.

EXERCISE 57.

LOWEST COMMON MULTIPLE.

84. When any number or expression is multiplied by something, the product is called a **Multiple** of the given number or expression. Thus, 10 is a multiple of 5, 50 is another multiple of 5, 100 still another multiple of 5, etc.; 10 is also a multiple of 2 as well as of 5, 50 is a multiple of 25 as well as of 5, etc.

1. Is 100 a multiple of 5? Is 100 a multiple of 10? Is 100 a multiple of 2?

2. Is $2a^2$ a multiple of 2? Of a? Of a^2?

3. Is ab^2x a multiple of a? Of b? Of x? Of ab? Of ax? Of abx?

85. From these questions it is evident that the same expression may be a multiple of several different expressions, in which case it is called a **Common Multiple** of those expressions.

4. Is ab^2x a common multiple of ab, ax and abx?

5. Is a^2bx also a common multiple of ab, ax and abx?

6. Is $a^2b^2x^2$ a common multiple of the same three expressions?

86. From what is here given, it is plain that two or more given expressions may have more than one common multiple. Indeed, if any common multiple of two or more given expressions be found, then if this common multiple be multiplied by any number whatever, the result will also be a common multiple of the given expression.

Any common multiple of two or more expressions contains all the prime factors of each of the given expressions.

That common multiple which contains the *least* number of prime factors is called the **Lowest Common Multiple** of the given expressions. The abbreviation L. C. M. is frequently used to stand for the lowest common multiple.

87. From what we have had it follows that to find the L. C. M. of two or more expressions we proceed as follows:

Resolve each expression into its prime factors and form a product in which each of these prime factors occurs as many times as it occurs in that one of the given expressions in which it occurs the greatest number of times.

Find the L. C. M. of $3a^2x^2y$ and $30ax^2$.

$$3a^2x^2y = 3aaxxy;$$
$$30ax^2 = 3\times 2\times 5axx.$$

The prime factors are 2, 3, 5, a, x, y. The prime factor 2 occurs once in the second expression, hence it occurs once in the L. C. M. Similarly, the prime factors 3 and 5 occur once in the L. C. M. The prime factor a occurs once in the second expression, but twice in the first expression; hence it will occur twice in the L. C. M. Similarly, the prime factor x occurs twice in the L. C. M, Finally, the prime factor y occurs once in the L. C. M. Collecting results, we see that the L. C. M. is equal to $2\times 3\times 5\ a\ a\ x\ x\ y$, or $30a^2x^2y$.

Find the L. C. M. of the following expressions:

1. $14x^2y^2$ and $42xy^3z$.
2. $7xyz$ and $8ax^2y^3z^4$.
3. $9abc$ and $15a^2b^3x^5y$.
4. $17a^2b^2c^2$ and $17a^3b^3c^3$.
5. $27x^4y^5z^6$ and $2m^2x^3y$.
6. $14x^2y^2$, $9abc$, and $7xyz$.
7. $42xy^3z$, $8ax^2y^3z^4$, and bw^5.
8. $a^2b^2c^2x^3y^3z^3$, $abcu^4v^4w^4$, and $uvwxyzabc$.
9. $5ax$, $10ay$, $25b^2z^2$, $100a^2c^2$, and $50abcxy^2z$.
10. $5rst$, 12, rt, 30, $20rt^2$, $15st^3$, and $4s^2t$.
11. $14c^4d^6$, $21ac^2$, $70a^2bc$, $105ad^6$, and $35bcd^7$.
12. $2bcd^7$, $15ad^4$, $14b^3c^5$, $10b^3d^6$, $21a^2c^4$, and $35bd$.
13. $5ac^4$, $10ac^5$, $7d^6$, $6a^2b^2$, $15b^3c^3$, and $14ad^7$.
14. $30a^2b^2c^3$, $70ac^4d^7$, $42a^2b^3d^6$, $105b^3c^5d^4$, and $35c^4d^5$.
15. $20xy^2z$, $45xz^2v^4$, $36\ zu^4v^3$, $3xyzu^4v^3$, and $5\ u^3v^5$.
16. $60zv$, $45xz^3v$, $90y^2zuv$, $9xzu^4v$, and $30xv^5$.

17. Multiply the L. C. M. of $3a^2xy$ *and* $2ax^2y^2$ by the H. C. F. of the same expressions.

18. Divide the product of a^2bx and ab by the H. C. F. of a^2bx and ab, and compare the quotient with the L. C. M. of a^2bx and ab.

19. Divide the L. C. M. of $2a^2x^3y$ and $3\,ax^2y$ by the first of the two expression, and compare the result with quotient obtained by dividing the second of the two expressions by the H. C. F. of the two expressions.

20. Divide the product of $5a^2x^3y$ and $15abx^4z$, by the L. C. M. of the same two expressions, and compare the quotient with the H. C. F. of the same two expresssions.

EXERCISE 58.

FRACTIONS.

88. We have already used the fractional form $\frac{a}{b}$ as another way of writing $a \div b$, so that $\frac{a}{b}$ is an expression of division. We have already learned that in any case of division the divisor multiplied by the quotient equals the dividend, or, in the language of fractions, precisely the same thing may be written,

$$Denominator \times Quotient = Numerator.$$

89. From this equation it is plain to see that if the denominator remains unchanged, multiplying the numerator by any number multiplies the quotient by the same number, and dividing the numerator by any number divides the quotient by the same number, or, as it is more often stated, *multiplying the numerator by any number multiplies the fraction by that number*, and *dividing the numerator by any number divides the fraction by that number.*

90. Again, from the same equation,

Denominator × *Quotient* = *Numerator*,

it is also plain that if the numerator remains unchanged, multiplying the denominator by any number divides the quotient by that number, and dividing the denominator by any number multiplies the quotient by that number, or, as it is more often stated, *multiplying the denominator by any number divides the fraction by that number*, and *dividing the denominator by any number multiplies the fraction by that number.*

91. Once more, from the same equation,

Denominator × *Quotient* = *Numerator*,

it is plain that if the quotient remain unchanged, multiplying the denominator by any number multiplies the numerator by the same number, or, stated in another way, *multiplying both numerator and denominator by the same number does not alter the value of the fraction.* It is also evident from the same equation that dividing both numerator and denominator by the same number leaves the quotient unchanged, or, stated in the usual form, *dividing both numerator and denominator by the same number does not alter the value of the fraction.*

92. When the numerator and denominator of a fraction are each divided by all the factors common to both numerator and denominator so that the resulting numerator and denominator contain no common factor, the fraction is then said to be in its **Lowest Terms.**

Of course, then, to reduce a fraction to its lowest terms we *divide both numerator and denominator by every factor common to both, i. e., by the H. C. F. of the numerator and denominator.*

Reduce the following fractions to their lowest terms:

1. $\frac{3ax}{12bx^2}$.

2. $\frac{32c^2x^2}{48bcx}$.

3. $\frac{12abc}{15cde}$.

4. $\frac{34x^4y^3z^2}{39xz^5}$.

5. $\frac{25m^3r^4x^{10}}{10m^5r^2x^8}$.

6. $\frac{39abx}{169axy}$.

7. $\frac{143a^6b^5x^4}{187a^{11}x^9y^5}$.

8. $\frac{209r^3s^2t^4v}{114s^3tv^4}$.

9. $\frac{300x^4y^3z}{301a^3b^2z}$.

10. $\frac{300x^4y^3zu^4}{301a^3b^2cu}$.

EXERCISE 58.

ADDITION OF FRACTIONS.

93. If two or more fractions have the same denominator, the fractions may be added by adding the numerators and placing the sum over this common denominator; but if the denominators are not the same we must multiply the numerator and denominator of each fraction by such a number as will make all the denominators the same, and *then* add the numerators, and place the sum over this common denominator.

94. The process of changing the numerators and denominators of fractions, so that each fraction shall preserve the same value it had before, while the denominators are all made alike is called **Reducing to a Common Denominator.**

For example, suppose we wish to add $\frac{ab}{m^2}$ and $\frac{4x}{mn}$ together. We must reduce to a common denominator, which, of course, must be a common multiple of m^2 and mn. *Any* common multiple will do, but the *lowest* common multiple is preferable.

The lowest common multiple is plainly m^2n; hence, we must multiply the numerator and denominator of the first fraction by n, and the numerator and denominator of the second fraction by m. We then have

$$\frac{4abn}{m^2n}+\frac{4mx}{m^2n}.$$

Plainly, then, the sum of the two fractions is

$$\frac{4abn+4mx}{m^2n}.$$

Hence, we may write the equation,

$$\frac{ab}{m^2}+\frac{4x}{mn}=\frac{4abn+4mx}{m^2n.}$$

If we had three fractions to add together, would the common denominator be a common multiple of each of the given denominators? Would there be any preference for the *lowest* common denominator? Why?

If *any number* of fractions are to be added together, what would you prefer to take for a common denominator?

State the method to be pursued when two or more fractions are to be added together.

Add the following fractions:

1. $\frac{2a}{x}$ and $\frac{3a}{2x}$.
2. $\frac{ab}{m}$ and $\frac{x}{m^2n}$.
3. $\frac{cd}{x^2y}$ and $\frac{cd}{xy^2}$.
4. $\frac{abc}{xyz}$ and $\frac{ab^2c^3}{xy^2z^3}$.
5. $\frac{m^2r^2}{5xy}$ and $\frac{r^2s^2}{15x^3}$.
6. $\frac{4ab}{17cd}$ and $\frac{4cd}{17ab}$.
7. $\frac{6ab^2}{12xy}$ and $\frac{7z}{4xz^2}$.
8. $\frac{16rst}{17uvw}$ and $\frac{16rst}{17xyz}$.
9. $\frac{10a^2b^6}{5ab^7}$ and $\frac{7}{ab}$.
10. $\frac{12uv}{13st}$ and $\frac{13st}{12uv}$.
11. $\frac{a}{b}$, $\frac{x}{3b^2}$, $\frac{y^2}{m^2}$, $\frac{4uv}{3a^2b}$, and $\frac{12}{a^2b^2m^2}$.
12. $\frac{x^2}{y^2}$, $\frac{m^3}{n^3}$, $\frac{u}{v}$, $\frac{14xy}{y^3}$, and $\frac{7}{u^3v}$

EXERCISE 59.

SUBTRACTION OF FRACTIONS.

If one fraction is to be subtracted from another and the fractions have the same denominator, we may subtract the second numerator from the first and place the remainder over this common denominator; but if the denominators are different, we must first reduce the fractions to a common denominator and then perform the subtraction.

1. From $\frac{3x}{2a}$ take $\frac{4x}{3a^2}$.
2. From $\frac{3a^2x}{2b}$ take $\frac{4cx}{3b^2}$.
3. From $\frac{15abc}{xyz}$ take $\frac{xyz}{15abc}$.
4. From $\frac{10a^2bc}{15cde}$ take $\frac{12ab}{20c^2d}$.
5. From $\frac{a^2+b^2}{1}$ take $\frac{1}{a^2-b^2}$.
6. From $\frac{abc^2}{bcd^3}$ take $\frac{cde^2}{def^3}$.
7. From $\frac{12xy}{7z}$ take $\frac{12xy}{7xyz}$.
8. From $\frac{3ab}{9xyz^2}$ take $\frac{12xz}{16xz^2u}$.
9. From $\frac{17a}{51mr}$ take $\frac{6x}{9m^2r^3}$.
10. From $\frac{12mr^2}{20m^2r}$ take $\frac{14ns^2}{21n^2s}$.

EXERCISE 60.

MULTIPLICATION OF FRACTIONS.

95. We are to multiply $\frac{a}{b}$ by $\frac{c}{d}$.

Now to multiply by $\frac{c}{d}$ means to multiply by c, and divide the result by d. Therefore to multiply $\frac{a}{b}$ by $\frac{c}{d}$ we first multiply $\frac{a}{b}$ by c, and then divide the result by d.

We may multiply a fraction by multiplying the numerator; hence, $\frac{a}{b} \times c = \frac{ac}{b}$.

We may divide a fraction by multiplying the denominator; hence, $\frac{ac}{b} \div d = \frac{ac}{bd}$.

Therefore, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Hence to multiply two fractions together *multiply the numerators together for a new numerator, and the denominators together for a new demoninator.*

96. Suppose we wish the product of three fractions, as for instance, $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f}$.

We may multiply the first two together, as just explained, giving $\frac{ac}{bd}$,

and we can then multiply this result by the third fraction, by the method just explained, giving

$$\frac{ace}{bdf}.$$

Therefore, $\frac{a}{b} \times \frac{c}{d} \times \frac{e}{f} = \frac{ace}{bdf}$.

As this can be extended to any number of fractions, we have:

The product of any number of fractions is found by multiplying all the numerators together for a new numerator, and all the denominators together for a new denominator.

Multiply the following fractions:

1. $\frac{3ab}{4x^2}$ and $\frac{5cd}{6a^2x}$.

2. $\frac{4xy}{6uv}$ and $-\frac{8uvw}{12xyz}$.

3. $\frac{a^2x^2}{b^2y^2}$, $-\frac{bxy}{cz^2}$, and $\frac{6byz}{7ax}$.

4. $-\frac{nr}{uv}$, $-\frac{u^3x}{wx^4}$, and $\frac{v^3}{xz^5}$.

5. $\frac{4rstu^2v}{5uvxy}$, $\frac{10x^2y}{16u^3v^2t}$, and $\frac{2abu^2v^2}{rsx}$.

6. $\frac{143nrs}{187xyz}$, $\frac{34x^3z}{39wt}$, and $\frac{2w^2}{3s}$.

7. $\frac{5ab}{6xy}$, $\frac{7cd}{8uv}$, and $\frac{9ef}{70w}$

8. $-\frac{a^3}{x^3}$, $-\frac{a^3}{x^3}$, $-\frac{a^3}{x^3}$, and $-\frac{a^3}{x^3}$.

9. $\frac{a^2b^2c^2}{1}$, $\frac{2}{uvw}$, and $\frac{ua}{a^2bc}$.

10. $\frac{a}{x}$, $-\frac{b}{y}$, $-\frac{bx}{ay}$, and $\frac{y^2}{b^2}$.

11. $-\frac{a^4}{x^4}$, $-\frac{x^2}{y}$, $\frac{y^3z}{z^2w^3}$, and $-\frac{w^2xyz}{abc}$.

12. $\frac{abc}{xyz}$, $\frac{a^2x}{b^2z}$, $-\frac{bz^3}{xy^2}$, and $-\frac{cy}{xy^2z^3}$.

13. $\frac{2rst^3}{4xyz^4}$, $\frac{42xz^2}{14y^3}$, $\frac{3mrx}{27yz}$, and $\frac{19r^2x}{38rxz^3}$.

14. $\frac{5uvw}{12xyz}$, $-\frac{18uyz^3}{15vw^2y}$, and $\frac{17uvxy}{85vwz^4}$.

15. $\frac{16abrst^6}{17uvn^5xy}$, $-\frac{51avs^4x}{64bur^2wy}$, and $\frac{16awx^2z}{9uvyz^3}$.

16. $-\frac{m^2n^3r^4s^5}{xyz}$, $\frac{5nx^4ry^2}{7cz^4}$, and $-\frac{m^4r^5}{9v}$.

17. $\frac{16ax}{17by^3}$, $-\frac{4cz}{7wv^3}$, $-\frac{34xy}{6vw}$, and $-\frac{21a^4z}{4wy^2}$.

18. $\frac{axyz}{buvw}$, $-\frac{b^2u^2x^2}{r^2y^2z^2}$, $\frac{w^5x}{yz^4}$, and $\frac{tvw}{xyz}$.

19. $\frac{abk}{uvx^2}$, $\frac{mr}{u^2}$, $-\frac{r^2kv}{wx^3}$, and $\frac{16mu}{5xyz}$.

20. $\frac{ax}{by}$, $\frac{b^2z}{c^2y}$, $-\frac{4wx}{mz^2}$, and $-\frac{rxy}{sy^2z}$.

EXERCISE 61.

Division of Fractions.

97. We are to divide $\frac{a}{b}$ by $\frac{c}{d}$.

We may write the quotient in the form of a fraction, where the numerator is itself the fraction $\frac{a}{b}$ and the denominator is the fraction $\frac{c}{d}$.

Hence $$\frac{\frac{a}{b}}{\frac{c}{d}} = \text{quotient.}$$

Let us multiply both numerator and denominator of this fraction by bd. We know this will not change the value of the quotient.

$$\frac{a}{b} \times bd = \frac{abd}{b} = ad,$$

$$\frac{c}{d} \times bd = \frac{bcd}{d} = bc.$$

Hence $$\text{quotient} = \frac{ad}{bc}.$$

Therefore $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

This result may be obtained by *multiplying* the fraction $\frac{a}{b}$ by the fraction $\frac{d}{c}$, which last is the divisor inverted.

Hence, to divide one fraction by another, *invert the terms of the divisor and multiply.*

1. Divide $\frac{2x}{3y}$ by $\frac{4z}{5v}$.

2. Divide $\frac{rst}{uvw}$ by $-\frac{2uv}{3sw}$.

3. Divide $-\frac{2mr}{6tx}$ by $\frac{9r^2t}{7x^2}$.

4. Divide $\frac{34x^2y}{51uv}$ by $\frac{2y^2z}{3vw^2}$.

5. Divide $-\frac{10mk^2}{6rs^5}$ by $-\frac{39ks^4}{52x^2z}$.

6. Divide $\frac{1}{5kst}$ by $\frac{1}{10k^2xz}$.

7. Divide the sum of $\frac{8}{x}$ and $\frac{3x}{x^2}$ by the sum of $\frac{6x^2}{z^2}$ and $\frac{x^2yz}{yz^3}$.

8. Divide the sum of $\frac{6x^2}{z^2}$ and $\frac{5x^2yz}{yz^3}$ by the product of $\frac{8}{x}$ and $\frac{3x}{x^2}$.

9. Divide $\frac{x^3}{y^3}$ by $\frac{uv}{z^2x}$, and divide the result by $\frac{m^2n}{r^2s}$.

10. Divide $\frac{2}{x}$ by $\frac{ab^2}{c}$, and multiply the result by $\frac{1}{4acx}$.

11. Divide $\frac{a^2}{x}$ by $\frac{rst}{xyz}$, and multiply the quotient by $\frac{uvx^3}{a^2y^2}$.

12. Divide $\frac{m}{s^2}$ by $\frac{rt}{uv^3}$, and divide the quotient by $-\frac{1}{r^2st}$.

13. Divide $\frac{a}{x^2}$ by $-\frac{uvw}{rst^3}$, and multiply the quotient by the quotient of $\frac{u}{v} \div \frac{s}{t}$.

14. Multiply mr^2x^2 by the quotient of $\frac{u}{v^2} \div \frac{w^3}{rx}$.

15. Add the quotient of $\frac{axy}{uvw^3} \div \frac{uxw}{a^2rt}$ to the quotient of $\frac{axw}{rty} \div \frac{uvx}{sxt}$.

16. Add the product of $\frac{rs^2x^2}{u^2vw^3}$ and $\frac{uxw}{r^2v}$ to the quotient of $\frac{4rx^2}{7uvy} \div \frac{17uxw}{2vy^3}$.

17. Subtract the quotient of $\frac{rst}{uvw} \div \frac{mrx^3}{v^2w^2y}$ from the product of $\frac{uvx}{abc}$ and $\frac{r^2sy}{u^2xz}$.

18. Divide $\frac{r^3s}{x^2y}$ by $\frac{myz}{rsx}$, and divide the result by $-\frac{1}{axz}$ and divide *this* result by $\frac{rsx^2}{u^2vy}$.

19. Divide $\frac{a^2x}{b^2yz}$ by 3, divide the result by $\frac{by}{4x^3}$, and divide this result by the product of $\frac{a}{x^2}$ and $\frac{u^2v}{wy}$.

20. Multiply $\frac{1}{rsty}$ by $\frac{abx^2}{uvy^2}$ and divide the product by the product of $\frac{1}{x^2y}$ and $\frac{1}{a^2bz}$.

CHAPTER X.

SIMPLE EQUATIONS.

EXERCISE 62

DEFINITIONS AND GENERAL PRINCIPLES.

98. An **Equation** is the statement of equality which exists between two expressions.

99. The **Members** or **Sides** of an equation are the parts on either side of the sign $=$, and are distinguished as the **First** and **Second Members**, or **Left** and **Right Sides**, respectively.

Students often have a careless habit of calling almost everything in Algebra an *equation*. Thus, we hear $a^2+2ab+b^2$ called an equation instead of an expression or a trinomial. It is better to call an expression an expression, and an equation an equation. It is also better to say "multiply *both members* of the equation by 2," than "multiply *the equation* by 2," etc.

100. If the two sides of an equation are equal, no matter what numbers be substituted for the letters, the equation is called an **Identical Equation** or simply an **Identity**. Thus, the following equations are identities:

$$(x+a)(x+a)=x^2+2ax+a^2,$$
$$x(x-a)=x^2-ax,$$
$$(x+a)(x-a)=x^2-a^2,$$

for the equations are true, no matter what numbers be put for x and a.

101. If the two sides of an equation are equal only when a particular number or numbers are substituted for

the letters, the equation is called a **Conditional Equation** or simply an equation. The following are conditional equations: $x+1=2$,

$$x+7=2x+4,$$

for the first equation is only true when $x=1$ and the second is only true when $x=3$.

102. A letter for which a particular number must be substituted, in order that the two sides of the equation may be equal, is called an **Unknown Number.**

103. A number which, when substituted for the unknown number, makes the two sides of the equation equal, is said to **Satisfy** the equation, and is called the **Root of the Equation** or a **Value of the Unknown Number.**

104. To **Solve** an equation is to find the root or roots.

105. A **Simple Equation** or **Equation of the First Degree**, is one which contains only the *first power* of the unknown number. Thus,

$$8-\tfrac{1}{2}x-4x=\frac{x+3.}{4}$$

is a simple equation, but $x^2+4x=5$ is not a simple equation.

106. The axioms which are useful in the solution of simple equations are as follows:

I. If we add to equals the same number or equal numbers, the sums will be equal.

II. If we take from equals the same number or equal numbers, the remainders will be equal.

III. If we multiply equals by the same number or equal numbers, the products will be equal.

IV. If we divide equals by the same number or equal numbers, the quotients will be equal.

107. The first two of these axioms are embodied in the **Principle of Transposition**: *Any term in one member of an equation may be transposed to the other member, provided its sign be changed.*

108. The use of the axioms is illustrated by the following examples:

(1) Solve the equation $\frac{x}{5}=6$.

Multiplying both sides of the equation (axiom 3) by 5, we get $x=30$.

(2) Solve the equation $\frac{3x}{7}=9$.

Multiplying both sides of the equation (axiom 3) by 7, we have $3x=63$.

Dividing both sides of the equation (axiom 4) by 3, we obtain $x=21$.

(3) Solve $1+\frac{x}{2}+\frac{x}{3}=x-7$.

Multiplying both sides of the equation (axiom 3) by 6, we have $6+3x+2x=6x-42$.

Transposing (axioms 1 and 2) unknown numbers to the left side and known numbers to the right side, we get

$$3x+2x-6x=-42-6.$$

Uniting similar terms, $-x=-48$.

Dividing both sides (axiom 4) by -1, we have

$$x=48.$$

(4) Solve $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=2-\frac{x}{6}+\frac{5x}{12}$.

Multiplying both sides of the equation (axiom 3) by 12, we have $6x-4x+3x=24-2x+5x$.

Transposing unknown numbers to left side,

$$6x-4x+3x+2x-5x=24.$$

Uniting similar terms, $2x=24$.

Dividing both sides (axiom 4) by 2, $x=12$.

109. The object in multiplying both sides of an equation by the same number is to **Clear the Equation of Fractions.** This may be accomplished in two ways: First, *multiply both members of the equation by the product of the denominators of all the fractions.*

Or, if we prefer, we may *multiply both members of the equation by the least common denominator of all the fractions.*

Thus, solve the equation $\frac{3x}{4}+\frac{7x}{15}+\frac{11x}{6}-366=0.$

Multiplying both sides by 60, the least common denominator of the fractions, we get

$$45x+28x+110x-21960=0.$$

Transposing known quantity to right side and uniting similar terms, we have $183x=21960.$

Dividing both members by 183, we obtain

$$x=120.$$

110. In clearing an equation of fractions the student must take special care when a minus sign occurs before a fraction and the numerator is not a monomial. It must be remembered that *the dividing line of a fraction is the same as a parenthesis enclosing the numerator.* Thus, $2-\frac{x+4}{1}$ is the same as $2-(x+4)$, and $2-\frac{x+4}{5}$ is the is the same as $2-\frac{1}{5}(x+4)$. We will illustrate this point by a few examples.

(1) Solve $\frac{x+2}{3}=\frac{5-x}{4}-\frac{x+1}{2}+14.$

Multiply both sides by 12,

$$4(x+2)=3(5-x)-6(x+1)+168.$$

Perform the indicated operations,

$$(4x+8)=(15-3x)-(6x+6)+168.$$

Remove parentheses,

$$4x+8=15-3x-6x-6+168.$$

Transposing the known numbers to the right side and the unknown numbers to the left side,

$$4x+3x+6x=15-6+168-8.$$

Uniting similar terms,

$$13x=169;$$

whence $$x=13.$$

(2) Solve $x-\frac{3x-2}{5}=3-\frac{2x-5}{3}$.

Multiply both sides by 15,

$$15x-3(3x-2)=45-5(2x-5).$$

Perform indicated operations,

$$15x-(9x-6)=45-(10x-25).$$

Remove parentheses,

$$15x-9x+6=45-10x+25.$$

Transpose the known numbers to the right side and the unknown numbers to the left side,

$$15x-9x+10x=45+25-6.$$

Uniting similar terms, $16x=64$;

whence $$x=4.$$

111. *The signs of all the terms in the numerator of each fraction preceded by the minus sign must be changed when the equation is cleared of fractions.* The neglect of this is *a very common source of error.*

112. We will now formulate the **Method for Solving any Simple Equation.**

I. Clear the equation of fractions.

II. Transpose the known numbers to the right side and the unknown numbers to the left side of the equation.

III. Unite similar terms.

IV. Divide both sides by the coefficient of the unknown quantity.

This is generally the best order to pursue, although it is sometimes shorter to unite similar terms before clearing of fractions.

EXERCISE 63.

EXAMPLES.

Solve the following equations :

1. $\frac{x}{5}+8=13.$

2. $\frac{5}{6}=\frac{3}{4x}.$

3. $\frac{12}{x}+5=8.$

4. $\frac{x}{2}+\frac{x}{3}=5.$

5. $\frac{x}{3}+\frac{1}{6}=\frac{x}{2}.$

6. $\frac{x}{2}+7=x-\frac{x}{3}.$

7. $\frac{x}{3}+\frac{x}{5}=8.$

8. $\frac{3}{4}x+\frac{5}{6}x=38.$

9. $\frac{x}{2}-\frac{4x}{9}=5.$

10. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=7.$

11. $\frac{x}{4}+\frac{x}{5}=9.$

12. $\frac{2}{3}y+\frac{4}{5}y=22.$

13. $\frac{3x}{4}-\frac{2x}{7}=13.$

14. $\frac{y}{5}+\frac{3y}{7}-\frac{y}{2}=9.$

15. $13x-\frac{8x}{9}+\frac{7x}{2}=15x+22$

16. $\frac{2z}{5}-\frac{3z}{2}=\frac{z}{2}-32.$

17. $2x-\frac{3}{5}x=\frac{3x}{2}-\frac{1}{2}-\frac{2x}{5}+2.$

18. $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}-\frac{x}{6}+\frac{x}{8}+\frac{x}{12}=11.$

In the following it will secure greater accuracy if the student will put the work down in full, as in Art. 110. Thus, in the next example the fraction $-\frac{5x+4}{3}$ must be multiplied by 6, and it is better to write the result in the form $-2(5x+4)$ first, and afterwards in the form $-10x-8$, *in order to avoid mistakes in the signs.*

19. $\frac{x}{3}-\frac{5x+4}{3}=\frac{4x-9}{3}$.

20. $x-\frac{x-2}{3}=\frac{x+23}{4}-\frac{10+x}{5}$.

21. $\frac{3x+9}{4}=\frac{5x+16}{7}$.

22. $\frac{5-x}{7}=\frac{11-3x}{23}$.

23. $\frac{5x-6}{5}-\frac{3x}{13}=\frac{x-4}{9}$.

24. $\frac{12-3x}{4}-\frac{3x-11}{3}=1$.

25. $\frac{2(7x-10)}{3}-20=\frac{50-x}{2}$.

26. $\frac{x-5}{4}-\frac{284-x}{5}-6x=0$.

27. $\frac{2x+1}{3}+\frac{3x+1}{4}=28-\frac{5x-2}{7}$.

28. $\frac{5x-6}{7}-\frac{3x-4}{9}-\frac{4x}{11}=0$.

29. $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$.

30. $\frac{5x-11}{4}-\frac{x-1}{10}=\frac{11x-1}{12}$.

31. $\frac{8-x}{6}+\frac{3x-5}{3}=\frac{x+6}{2}-\frac{x}{3}$.

32. $\frac{x+3}{2}-\frac{1}{3}(x-2)=\frac{1}{12}(3x-5)+\frac{1}{4}$.

EXERCISE 64.

LITERAL EQUATIONS.

113. If the known numbers in an equation are represented by letters instead of by figures, the equation is spoken of as a **Literal Equation.** Of course no different principles hold in the solution of such equations, than in the solution of numerical equations.

It must be remembered that the first and intermediate letters of the alphabet stand for numbers supposed to be known or given.

Solve the following literal equations:

1. $6mx+2a=8mx-2b.$

Transposing the unknown numbers to the left side, and the known numbers to the right side, we obtain

$$6mx-8mx=-2b-2a..$$

Uniting similar terms

$$-2mx=-2b-2a.$$

Dividing both sides by $-2m$

$$x=\frac{a+b}{m},$$

which is the value of x.

2. $\frac{ax-b}{c}-\frac{bx+c}{a}=abc.$

Multiply both sides by ac

$$a(ax-b)-c(bx+c)=a^2bc^2.$$

Remove the parentheses

$$a^2x-ab-bcx-c^2=a^2bc^2.$$

Transposing known numbers to right side

$$a^2x-bcx=a^2bc^2+ab+c^2.$$

Uniting with a parenthesis

$$(a^2-bc)x=a^2bc^2+ab+c^2.$$

Dividing both sides by (a^2-bc)

$$x=\frac{a^2bc^2+ab+c^2}{a^2-bc}.$$

3. $x+a=b.$
4. $3x+6n=9r.$
5. $a+b-x=0.$
6. $x+a-b=c.$
7. $y-3+a=8.$
8. $mx=r.$
9. $mx+p=r.$
10. $18+ax-b=2x.$
11. $x+5x-b=2a.$
12. $3a+2y-4b=5y-b.$
13. $6a-8b+6ny=4a-6b-2c+8ny.$
14. $5mx-9a-11b=3mx-4b+7c-5mx-2a.$

15. $12(x-c)=7(c+x)$.
16. $3(a-5x)=4(3x-2c)$.
17. $7(a-x)=6(b-x)$.
18. $(b-1)y=b-y$.
19. $(1-b)x=a-x$.
20. $5(5c-2x)=4x-6c$.
21. $a(bx-c)=ac-abx$.
22. $nx-r=x$.
23. $ad-dy=my-am$.
24. $p(x-1)+x=q-p$.
25. $r(1+x)+n(r+p)=rx+np+x$.
26. $nr-n(y+1)+y=n(2-y)$.
27. $(b+1)x+ab=b(a+x)+a$.
28. $\frac{3b(x-a)}{5a}+\frac{(x-b^2)}{15b}=-\frac{(4a+cx)b}{6a}$.
29. $\frac{nx}{r}-\frac{r-x}{2p}+\frac{n(r-x)}{3q}=n$.
30. $\frac{2ax-2b}{3b}-\frac{ax-a}{2b}=\frac{ax}{b}-\frac{2}{3}$.
31. $\frac{b+c}{x}+a=\frac{a}{x}-b+\frac{a(a-1)-b(b-1)+c}{x}$.

EXERCISE 64*a*.

SYMBOLIC EXPRESSIONS.

114. In solving a problem in Algebra we must not only select some letter to stand for the unknown number, but we are required to find expressions which will symbolize all other numbers which occur in the problem. Such may be called **Symbolic Expressions.** Thus, if the sum of two numbers is 100, and if x stands for one of them, then the expression $100-x$ stands for the other number. If x is the price of one horse, then $10x$ stands for the cost of 10 horses; if 5 yards of cloth cost x dollars, then the cost of one yard is represented by the expression $\frac{x}{5}$, etc. Some drill in the formation of symbolic expressions will be of help in the solution of problems.

1. The sum of two numbers is 85. The first number is 8, what is the second? The first number is n, what is the second?

2. The sum of two numbers is a. The first number is 5, what is the second? The first number is b, what is the second?

3. A train travels at the rate of 20 miles per hour for 3 hours; how far does it go? A train travels at the rate of r miles per hour for 3 hours; how far does it go? A train travels at the rate of r miles per hour for t hours; how far does it go?

4. What must be added to 100 to make a? What must be added to x to make a?

5. One factor of 100 is 10; what is the other? One factor of 100 is x; what is the other?

6. What *two* numbers differ from 100 by 7? What two numbers differ from 100 by x? What two numbers differ from n by x?

7. How much will n apples cost at c cents apiece?

8. If one apple costs 2 cents, how many can you get for x cents? If one apple costs c cents, how many can you get for x cents? How many can you get for d dollars?

9. How many hours will it take to go x miles at 4 miles per hour? How many hours will it take to go x miles at r miles per hour? How many *minutes* will it take?

10. A train goes 150 miles in h hours; what is the rate per hour of the train? A train goes m miles in h hours; what is the rate of the train?

11. The rate of a train is r. How far will it go in time t?

12. What is the interest on d dollars for t years, at 5 per cent.? What does the principal and interest amount to for this time? What is the interest on d dollars for t years at r per cent.?

13. A man can do a piece of work in 10 days. How much of it can he do in one day? A man can do a piece of work in n days. How much of it can he do in one day?

14. A can do a piece of work in 9 days and B can do it in 12 days. What part can each do in one day? What part can both, *working together*, accomplish in one day? A can do a piece of work in a days and B can do it in b days. What part can both, working together, accomplish in one day?

15. A pipe will fill a cistern in 7 hours; what part runs in during one hour? Another pipe will fill the cistern in 5 hours; what part runs in during one hour? The cistern holds x gallons. How many gallons does each pipe carry in one hour?

16. One pipe will fill a cistern in a hours, and another pipe will fill it in b hours. What part does each pipe carry in one hour? What part do *both pipes* together carry in one hour?

17. The digit 5 stands in tens' place; what number is expressed? The digit, represented by x, stands in tens' place; what number is expressed? A digit represented by x stands in hundreds' place; what number is expressed?

18. If the first and second digits of a number are 5 and 7 respectively, what is the number? If the first and second digits of a number are represented by a and $a+2$ respectively, what is the number?

19. The three digits of a number beginning at the right are represented by x, $x+2$, and $x-3$; what is the number expressed by them?

20. Write three consecutive numbers of which 7 is the first. Write three consecutive numbers of which n is the first.

21. Write three consecutive *even* numbers of which 6 is the first. Write three consecutive even numbers of which $2n$ is the first.

22. Write three consecutive *odd* numbers of which 5 is the first. Write three consecutive odd numbers of which $2n+1$ is the first.

23. Write three consecutive numbers of which n is the greatest. Write three consecutive even numbers of which $2n$ is the greatest. Write three consecutive odd numbers of which $2n+1$ is the greatest.

115. It is well to note that if n stands for any whole number, then $2n$ is the symbolic expression for any even number, since it is exactly divisible by 2, and $2n+1$ is the symbolic expression for any odd number, since when divided by 2 the remainder is 1.

EXERCISE 64*b*.

PROBLEMS.

116. The student has already solved a sufficient number of problems to give him a general idea of the way problems are solved in Algebra. He will find his experience embodied in the following:

Directions for Solving Problems in Algebra.

I. Represent one of the unknown numbers, preferably the one whose value is asked for, by x.

II. Make symbolic expressions to represent each of the other unknown numbers mentioned in the problem.

III. Find, from the problem, two of these symbolic expressions that are equal to each other.

IV. Solve the equation thus formed.

Solve each of the following problems:

1. The sum of two numbers is 33 and their difference is 7. Find the numbers.

Let x= one of the numbers,
then $33-x$= the other number.
And because the difference of the two numbers is 7; therefore,

$$x-(33-x)=7.$$

Removing parenthesis $x-33+x=7.$
Transposing and uniting terms, $2x=40,$
whence $x=20.$
Therefore, one number is 20 and the other 13.

2. If 56 be added to a certain number, the result is treble that number. What is the number?

3. Divide 54 into two parts, such that $\frac{1}{4}$ of one part shall equal $\frac{1}{5}$ of the other.

4. Divide 100 into two parts, such that twice the smaller part shall exceed the larger part by 8.

5. Divide 10 into two parts, such that the sum of 3 times one part and 7 times the other part may be 42.

6. Divide 100 into two parts, such that the first divided by 2 shall be equal to the second divided by 2.

7. Find a number such that the sum of its half and its third may exceed the sum of its fourth and its fifth by 25.

8. The difference of two numbers is 14, and their sum is 52. Find the numbers.

9. George and Will have together 220 marbles. If George would give 14 to Will, they would have an equal number. How many marbles has each?

10. I bought a number of yards of cloth at 30 cents and an equal number of yards at 20 cents, paying $8.50 altogether. How many yards of each kind did I buy?

11. In a yard there are chickens and rabbits, and altogether they have 14 heads and 38 feet. How many rabbits and how many chickens in the yard?

12. I bought 3 books at a certain price, 5 books at double the price, and 4 books at treble the price of the first. In all I paid $15. What was the price of each kind of books?

13. A man is 32 years old and his son 7. In how many years will the father be just twice the age of his son?

14. A clerk saved $1280 in 8 years. The first three years he spent $360 per year; the next two years he spent $440 per year, and each year thereafter $40 more than the year preceding. What was his annual salary?

15. A man is 10 years older than his nephew, but 15 years ago he was twice as old as his nephew. Required the present age of each.

16. Two women went to market with equal amounts of money. One spent $1.20. The other spent $1.60 and returned home with just $\frac{1}{3}$ as much money as her companion. How much did each have at first?

17. Two farmers, A and B, owned a flock of sheep in common, which they agreed to divide. A took 132 sheep; B took 158 sheep and 9 lambs, three lambs having the value of one sheep, and paid A $127.20 in cash. What was the value of one sheep?

18. A man was engaged for 80 days under the agreement that he was to receive $4 for each day he worked, but was to forfeit $1.50 for each day he was idle. He received $276 in all. How many days was he idle?

19. A servant was told to buy a certain number of pounds of coffee. If she bought coffee at 26 cents a pound she would have 18 cents left, but if she took coffee at 30 cents a pound she would be short 10 cents. How many pounds of coffee was she told to buy, and how much money had she?

20. A number consists of two digits, the sum of the digits being 10. If the digits be reversed the new number is 18 larger than the original number. What is the number?

Let $x=$ the digit in tens' place.

Because the sum of digits is 10; therefore,

$10-x=$ the digit in units' place.

Hence the number expressed by these digits is

$$10x+10-x.$$

But the number expressed when the digits are reversed is

$$10(10-x)+x.$$

Because this is 18 larger than the original number; therefore,

$$10(10-x)+x-(10x+10-x)=18.$$

Removing parentheses

$$100-10x+x-10x-10+x=18.$$

Transposing terms $\quad -10x+x-10x+x=18-100+10.$

Uniting terms $\quad -18x=-72.$

Therefore, $\quad x=4$, one digit.

Therefore, $\quad 10-x=6$, the other digit.

Hence the number is 46.

21. A number consists of two digits, the second of which is double the first. If we reverse the digits the new number exceeds the original number by 36. What is the original number?

22. A number consists of two digits of which the sum is 12. If we subtract 18 from the number, we obtain the number with digits reversed. What is the number?

23. A man is 32 years old and his son 2. In how many years will the father be 4 times as old as his son?

24. A man is 32 years old and his son 8. In how many years will the father be 4 times as old as his son?

25. A man is 32 years old and his son 11. In how many years will the father be 4 times as old as his son?

26. A man is f years old and his son s. In how many years will the father be n times as old as his son?

27. How long will it take the sheriff, who can drive 7 miles an hour, to catch a thief who is going 5 miles per hour, and has a start of 4 hours?

28. A train leaves a station and travels at the rate of 25 miles per hour. Two hours later another train follows it, traveling at a rate of 35 miles per hour. How long before the second train will overtake the first?

29. The body A travels one yard a minute and is pursuing B, which is 10 yards ahead of it. If A moves 12 times as fast as B, how many minutes before A and B are together?

30. At what time between 2 and 3 o'clock are the hands of a clock together?

The minute hand moves over one minute space in one minute. At 2 o'clock the minute hand is pursuing the hour hand, which is 10 spaces ahead of it. The minute hand moves 12 times as fast as the hour hand.

31. At what time between 4 and 5 o'clock are the hands of a clock together?

32. A post has $\frac{1}{5}$ of its length in the earth, $\frac{3}{7}$ in the water and 13 feet in the air. How long is the post?

33. The distance around a triangle is 75 yards. The second side is $\frac{2}{3}$ and the third $\frac{5}{7}$ of the first side. What is the length of each side?

34. A man left half of his estate to his wife, $\frac{1}{10}$ for charity, $\frac{3}{8}$ to his children, and \$400 to his servants. What was the amount of his estate?

35. After a battle a general ascertained that only $\frac{2}{3}$ of his army was fit for service; $\frac{1}{5}$ of the men were wounded, and 2000 were dead or missing. What was the original strengh of his command?

36. I have three jars. The first contains $\frac{3}{16}$ and the second $\frac{5}{24}$ of the amount contained by the third jar. If I fill the first and pour the contents into the second, it still lacks 1 pint of being full. What is the capacity of each jar?

37. A mistress promised her servant \$150 a year and a new dress. The servant left at the end of 10 months, and received \$120 and the dress. What was the value of the dress?

38. A can do a piece of work in 50 days, B can do the same work in 60 days, and C can do the same work in 75 days. How many days will it take them to do the work together?

39. A cistern is filled by three pipes; the first pipe alone would fill it in 7 hours; the second in 9 hours, and the third in 5 hours. How long will it take to fill the cistern by the three pipes together?

40. The head of a fish weighs 2 pounds. The tail weighs as much as the head and half the body, and the body weighs as much as the head and tail together. What is the total weight of the fish?

41. A can dig a trench in $\frac{1}{2}$ the time it takes B, and B can do it in $\frac{2}{3}$ the time it takes C. By working together they do the work in 6 days. Required the time it would have taken each to have done it alone.

42. There is a group of 40 persons, consisting of men, women and children. The number of women is $\frac{3}{5}$ the number of men, and the number of children is $\frac{2}{3}$ the number of men and women together. How many each of men, women and children in the group?

43. A person had \$1000, part of which he loaned at 7 per cent. and the rest at 6 per cent.; the total interest received was \$63. How much was loaned at 7 per cent.?

Let x=the number of dollars loaned at 7 %.

Then $1000-x$=the number of dollars loaned at 6 %.

Therefore, $\frac{7x}{100}$=the interest received at 7 %,

and $\frac{6(1000-x)}{100}$=the interest received at 6 %.

Since the whole interest received was \$63; therefore,

$$\frac{7x}{100}+\frac{6(1000-x)}{100}=63;$$

whence $x=300$.

44. A man loaned a certain sum of money at $4\frac{1}{2}$ per cent. for 3 years; then he loaned the same money at $5\frac{1}{2}$ per cent. for 2 years; then at 5 per cent. for 4 years; then at 6 per cent. for 3 years. The principal and interest for the twelve years then amounted to \$3900. What was the original principal?

45. A principal of \$4800 having been loaned a certain number of years, amounts with simple interest to \$6972. During $\frac{1}{8}$ of the time it bore 7 per cent. interest; during $\frac{1}{4}$ of the time it bore $7\frac{1}{2}$ per cent., and during the remainder of the time it bore 8 per cent. For how long a time was the money loaned?

46. A man invested \$1000 at 4 per cent. interest, and 5 years afterwards he invested \$1800 at 5 per cent. In how many years will the two sums be equal?

47. A man invested \$1100 at 5 per cent. interest, and 5 years later he invested \$1000 at 7 per cent interest. How many years from this last date until the principal and interest of the two investments will equal each other?

48. A man invested a dollars at n per cent. and d years later he invested b dollars at r per cent. How long until the two sums, principal and interest, will be equal?

CHAPTER XI.

SIMULTANEOUS EQUATIONS.

EXERCISE 65.

DEFINITIONS AND GENERAL PRINCIPLES.

1. In the equation $x+y=25$ what is the valne of x if $y=4$? If $y=6$? If $y=3$? If $y=\frac{1}{2}$? If $y=\frac{3}{2}$?

In the same equation what is the value of y if $x=21$? If $x=19$? If $x=22$? If $x=24\frac{1}{2}$? If $x=23\frac{1}{2}$?

117. From this it is plain that in the equation $x+y=25$, the value of *either* x or y can easily be found *if the value of either is given*, and *neither* one of these can be found *unless* the value of the other is given.

It is also plain that there is an unlimited number of *pairs* of values which satisfy this equation $x+y=25$.

118. Equations in which an unlimited number of values can be found for the unknown numbers are called **Indeterminate Equations.**

2. In the equation $x+3y=33$ what is the value of x if $y=7$? If $y=5$? If $y=10$? If $y=4$? If $y=\frac{2}{3}$?

In the same equation what is the value of y if $x=12$? If $x=18$? If $x=3$? If $x=21$? If $x=31$.

Here as before there is an unlimited number of *pairs* of values of x and and y which satisfy the equation $x+3y=33$, and from this alone there is no preference of one pair of values over another pair.

119. The above illustrations are sufficient to show it is true in general that *the values of two unknown numbers cannot be determined from one equation containing two unknown numbers unless the value of one of them is given.*

3. If $4y-x=7$ and $x=5$, how would you find the value of y?

4. If $2y+3x=12$ and $2x=6$, how would you find the value of x? of y?

5. If $4x-5y=18$ and $5y=10$, how would you find the value of y? of x?

6. How much more is $x+2y$ than $x+y$? How much more is 28 than 25? If $x+y=25$ and if $x+2y=28$, what is the value of y? Knowing the value of y, how would you find x from $x+y=25$?

7. If $2x+y=18$ and if $3x+y=23$, what is the value of x? Knowing the value of x, how would you find y from $2x+y=18$?

8. If $5x-2y=22$ and $3x-2y=10$, what is the value of x? Knowing the value of x, how would you find the value of y from $3x-2y=10$?

9. If $4x-7y=1$ and $4x-5y=3$, what is the value of y? Knowing the value of y, how would you find x from $4x-5y=3$?

10. If $x=2y+4$ and $x=3y+2$, make an equation which will not contain the unknown number x. If $2y+4=3y+2$, find the value of y. Knowing the value of y, how would you find x from $x=2y+4$?

11. If $6x=5y+11$ and $6x=4y+12$, make an equation which will not contain the unknown number x. What value of y will satisfy the equation just found? Knowing the value of y, how can you find x from $6x=5y+11$?

12. If $3y=4x-11$ and $3y=x-2$, make an equation which will not contain the unknown number y. Solve the equation thus found. Knowing the value of x, how can you find y from $3y=x-2$?

120. In each of the above cases we have found that *the values of two unknown numbers can be found from two equations containing the two unknown numbers.*

121. When two or more equations containing several unknown numbers are so related that they are all satisfied simultaneously by the same values of the unknown numbers, the equations are called **Simultaneous Equations.**

122. Of course, if we can obtain from two equations containing two unknown numbers, a single conditional equation containing but one of the unknown numbers, the value of this unknown number can be found in the way explained in the last chapter. When from two equations containing two unknown numbers, we obtain one equation containing but one of the unknown numbers, we are said to **Eliminate** the other unknown number.

We will explain three methods of elimination: I. By Substitution. II. By Comparison. III. By Addition and Subtraction.

EXERCISE 66.

Elimination by Substitution.

123. Examples 2, 3 and 5 in the last exercise are illustrations of elimination of an unknown number by substitution. We give a few more examples worked by this method.

(1) Given $2x=6y-38$ and $y+23=3x$ to find x and y.

From the first of these equations the value of x in terms of y is found to be

$$x=3y-19. \quad (1)$$

Substituting $3y-19$ for x in the second of the given equations we get

$$y+23=9y-57. \quad (2)$$

Transposing $$y-9y=-57-23. \quad (3)$$

Uniting terms and dividing both sides by -1

$$8y=80, \quad (4)$$

whence $$y=10. \quad (5)$$

Substituting this value for y in (1) we get

$$x=30-19=11,$$

whence $x=11$ and $y=10$.

To verify, we substitute these values in the original equations and get

$$22=60-38 \text{ and } 10+23=33.$$

(2) Given $7x+3y=100$ and $3x-y=20$ to find x and y.

From the second equation the value of y in terms of x is readily seen to be

$$y=3x-20. \quad (1)$$

Substituting $3x-20$ for y in the first of the given equations, we find $$7x+9x-60=100, \quad (2)$$

whence by transposing and uniting terms

$$16x=160. \quad (3)$$

Therefore, $$x=10. \quad (4)$$

Substituting this value for x in (1) we have

$$y=30-20=10,$$

whence $x=10$ and $y=10$.

124. It is easy to see that the method of elimination used in these examples may be applied in the case of any system of simultaneous equations. Hence we learn how to eliminate an unknown number by the **Method of Substitution**:

From either equation express one of the unknown numbers in terms of the other, and substitute this value in the other equation.

EXERCISE 67.

EXAMPLES.

Solve the following by the method of substitution:

1. $x+4y=37.$
   $2x+5y=53.$

2. $x+5y=573.$
   $x+\ y=181.$

3. $2x-3y=100.$
   $2x+\ y=156.$

4. $3x+4y=253.$
   $y=5x.$

5. $2x+95=11y.$
   $x-3y=0.$

6. $5x-4y=6.$
   $8x=7y.$

7. $x=3y-19.$
   $y=3x-23.$

8. $x+y=347.$
   $x-y=153.$

9. $3x+2y=7.$
   $7x-\ y=5.$

10. $x+5y=47.$
    $x+\ y=15.$

11. $x+5y=35.$
    $3x+2y=27.$

12. $5x+7y=101.$
    $7x-\ y=55.$

13. $x=16-4y.$
    $y=34-4x.$

14. $2x=11+9y.$
    $3x-12y=15.$

15. $8x-\ y=22.$
    $2x-3y=0.$

16. $3x+4y=18.$
    $3x-2y=70.$

EXERCISE 68.

ELIMINATION BY COMPARISON.

125. We give a few examples of elimination of an unknown number by the method of comparison.

(1) Given $3x=73-y$ and $2x=y+32$, to find x and y.

From the first equation, $x=\frac{73-y}{3}$. (1)

From the second equation, $x=\frac{y+32}{2}$. (2)

Therefore $\frac{73-y}{3}=\frac{y+32}{2}$. (3)

Clearing of fractions, $2(73-y)=3(y+32)$. (4)

That is, $146-2y=3y+96$. (5)

Hence, by transposing and uniting terms,

$5y=50$, (6)

whence $y=10$. (7)

Then, from (1), $x=\frac{73-10}{3}=21$. (8)

Hence $x=21$ and $y=10$.

(2) Given $9y+8x=41$ and $11x-7y=37$, to find x and y.

From the first equation, $y=\frac{41-8x}{9}$. (1)

From the second equation, $y=\frac{11x-37}{7}$. (2)

Whence $\frac{41-8x}{9}=\frac{11x-37}{7}$. (3)

Clearing of fractions,

$7(41-8x)=9(11x-37)$. (4)

That is, $287-56x=99x-333$. (5)

Hence, by transposing and uniting terms,

$$155x=620. \qquad (6)$$

Therefore $x=4.$

Then, from (1), $y=\frac{41-32}{9}=1.$

Hence $x=4$ and $y=1.$

126. These examples are sufficient to teach us how to eliminate an unknown number by the **Method of Comparison:**

Express the same unknown number in terms of the other from each equation and equate the expressions thus found.

EXERCISE 69.

EXAMPLES.

Solve the following by the method of comparison:

1. $5x=63-8y.$
   $7x=39-3y.$

2. $2y=29-3x.$
   $5y=20+3x.$

3. $3x+y=31.$
   $5x=15+2y.$

4. $4x=19+7y.$
   $x+9y=37.$

5. $16y-3x=5.$
   $5x+28y=19.$

6. $x+y=24.$
   $x-y=16.$

7. $5x-3y=13.$
   $19x-5y=75.$

8. $7x+5y=60.$
   $13x-11y=10.$

9. $3x+2y=32.$
   $20x-3y=1.$

10. $11x-7y=37.$
    $8x+9y=41.$

11. $10x+9y=290.$
    $12x-11y=130.$

12. $x+y=579.$
    $x-y=333.$

EXERCISE 70.

ELIMINATION BY ADDITION AND SUBTRACTION.

127. The elimination in the following examples is done by the method of addition and subtraction.

(1) Given $x+y=579$ and $x-y=333$ find x and y.

Adding the left members and the right members of these two equations together, we obtain

$$x+y=579 \quad (1)$$
$$x-y=333 \quad (2)$$
$$2x=912 \quad (3)$$

whence $x=456$ (4)

Subtracting the members of (2) from the corresponding members of (1), we get

$$2y=246$$

whence $y=123$.

Therefore, $x=456$ and $y=123$.

(2) Given $15x-8y=30$ and $3x+2y=15$ to find x and y.

Multiplying both members of the second equation by 5, we have for the two equations

$$15x-8y=30 \quad (1)$$
$$15x+10y=75 \quad (2)$$

By subtraction $-18y=-45$ (3)

whence $y=2\frac{1}{2}$. (4)

Hence from (1) $15x-20=30$. (5)

Therfore $x=3\frac{1}{3}$. (6)

Or the value of x may be obtained in another way. Multiplying both members of the second of the given equations by 4, we have for the two equations

$$15x-8y=30 \quad (7)$$
$$12x+8y=60 \quad (8)$$

By addition $27x=90$ (9)

whence $x=\frac{90}{27}$ or $3\frac{1}{3}$.

(3) Given $11x+12y=100$ and $9x+8y=80$ to find x and y.

Multiplying both members of the first equation by 9, and both members of the second equation by 11, we obtain

$$99x+108y=900 \quad (1)$$
$$99x+\ 88y=880 \quad (2)$$

By subtraction $20y=\ 20$ (3)

whence $y=1$. (4)

Substituting this value for y in either of the original equations, we find $x=8$

whence $y=1$ and $x=8$

128. The above examples show us how to eliminate an unknown number by the **Method of Addition and Subtraction:**

Multiply both members of the equations by such numbers as will make the numerical coefficients of one of the unknown numbers the same in the resulting equations; then by addition or subtraction we can form an equation containing only the other unknown number.

Solve the following by the method of addition and subtraction:

1. $x+y=30$.
   $x-y=6$.

2. $5x+7y=176$.
   $5x-3y=46$.

3. $x+5y=573$.
   $x+\ y=181$.

4. $x+4y=37$.
   $2x+5y=53$.

5. $4x+3y=97$.
   $7x+3y=127$.

6. $24x+\ 7y=27$.
   $8x-33y=115$.

7. $2x+3y=41$.
   $4x+2y=54$.

8. $5x+7y=17$.
   $7x-5y=9$.

9. $7x-3y=27.$
$5x-6y=0.$

10. $23x+15y=4\frac{1}{4}.$
$48x+45y=18.$

11. $6x-7y=42.$
$7x-6y=75.$

12. $13y-7x=99.$
$9y-28x=52.$

13. $16x+17y=274.$
$24x-105y=150.$

14. $21x+8y=66.$
$28x-23y=13.$

EXERCISE 71.

Special Expedients.

129. The student will find that elimination by addition and subtraction is in most cases the shortest method. Occasionally, however, an example will be found which is more readily done in some one of the other ways. Sometimes, too, special expedients will still further abbreviate the processes. We give a few examples of this.

(1) Solve $3x+7y=29$, $7x+3y=41$.

By adding the members of the two equations, we obtain $10x+10y=70.$ (1)

By subtracting the members of second equation from those of the first, we have

$4x-4y=12.$ (2)

From (1) we get $x+y=7.$ (3)

From (2) we get $x-y=3.$ (4)

Whence, by addition and then by subtraction,

$x=5$ and $y=2.$

(2) Solve $\frac{3x}{4}-\frac{2y}{3}=1,\quad \frac{7x}{3}+\frac{5y}{6}=6.$

To solve these we must first clear each of fractions, giving $9x-8y=12$

and $14x+5y=36$

which can now be solved in any of the usual ways.

(3) Solve $\frac{9}{x}-\frac{4}{y}=1, \quad \frac{18}{x}+\frac{20}{y}=16.$

If these be cleared of fractions, the resulting equations will involve the product xy, and we would have equations of a kind we have not yet considered. But by considering $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown numbers, we may solve by the methods already used. For example, by multiplying both members of the first equation by 2, we get for the two equations

$$\frac{18}{x}-\frac{8}{y}=2 \qquad (1)$$

$$\frac{18}{x}+\frac{20}{y}=16. \qquad (2)$$

By subtraction $$\frac{28}{y}=14, \qquad (3)$$

whence $$\frac{2}{y}=1 \text{ or } y=2.$$

Therefore from (1) $$x=3.$$

We could have eliminated x by dividing both members of the second given equation by 2.

EXERCISE 72.

EXAMPLES.

Solve the following by any method :

1. $x+y=50.$
   $x-y=48.$

2. $2x+y=11.$
   $3x-y=4.$

3. $3x+4y=85.$
   $5x+4y=107.$

4. $3x+8y=19.$
   $3x-\ y=1.$

5. $3x+8y=59.$
   $6x+5y=107.$

6. $8x-15y=30.$
   $2x+\ 3y=15.$

7. $5x+8y=101.$
$9x+2y=95.$

8. $39x+27y=105.$
$52x+29y=133.$

9. $72x+14y=330.$
$63x+7y=273.$

10. $2x-7y=8.$
$4y-9x=19.$

11. $8x+3y=3.$
$12x+9y=3.$

12. $69y-17x=122.$
$14x-23y=31.$

13. $2x+7y-34=0.$
$5x+9y-51=0.$

14. $3y-4x-1=0.$
$18-3x=4y.$

15. $2x+3y=41.$
$3x+2y=39.$

16. $19x-21y=100.$
$21x-19y=140.$

17. $17x-13y=2.$
$13x+17y=236.$

18. $\frac{1}{2}x+\frac{3}{5}y=16.$
$\frac{3}{5}x+\frac{1}{2}y=17.$

19. $10x+\frac{y}{3}=210.$
$10y-\frac{x}{2}=290.$

20. $\frac{x}{7}+7y=251.$
$\frac{y}{7}+7x=299.$

21. $\frac{7x}{6}+\frac{5y}{3}=34.$
$\frac{7x}{8}+\frac{y}{8}=12.$

22. $\frac{2x}{3}+\frac{3y}{2}=16\frac{1}{6}.$
$\frac{3x}{2}-\frac{2y}{3}=16\frac{1}{6}.$

23. $\frac{1}{x}+\frac{1}{y}=6.$
$\frac{2}{x}-\frac{1}{y}=0.$

24. $\frac{3}{x}+\frac{8}{y}=3.$
$\frac{15}{x}-\frac{4}{y}=4.$

25. $\frac{1}{x}+\frac{2}{y}=10.$
$\frac{4}{x}+\frac{3}{y}=20.$

26. $\frac{7}{x}+\frac{5}{y}=19.$
$\frac{8}{x}-\frac{3}{y}=13.$

27. $4+y=\frac{3x}{4}$.
$x-8=\frac{4y}{5}$.

28. $\frac{x}{3}+\frac{y}{2}=\frac{4}{3}$.
$\frac{x}{2}+\frac{y}{3}=\frac{7}{6}$.

29. $\frac{x}{4}+\frac{y}{6}=3\frac{1}{2}$.
$\frac{x}{3}-\frac{y}{8}=\frac{1}{2}$.

30. $\frac{1}{x}+\frac{1}{y}=\frac{11}{30}$.
$\frac{1}{x}-\frac{1}{y}=\frac{1}{30}$.

31. $\frac{5}{3x}+\frac{2}{5y}=7$.
$\frac{7}{6x}-\frac{1}{10y}=3$.

32. $\frac{x+y}{8}+\frac{x-y}{6}=5$.
$\frac{x+y}{4}-\frac{x-y}{3}=10$.

33. $\frac{3x-5y}{2}+3=\frac{2x+y}{5}$.
$8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{4}$.

34. $\frac{4x+5y}{40}=x-y$.
$\frac{1}{2}-\frac{2x-y}{3}=2y$.

35. $\frac{5x-3}{4}-\frac{3x-19}{4}=2-\frac{3y-x}{6}$.
$\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16}$.

36. $\frac{7+x}{5}-\frac{2x-y}{5}=3y-5$.
$\frac{5y-7}{2}-\frac{6x-3}{6}=18-5x$.

EXERCISE 73.

SIMULTANEOUS EQUATIONS CONTAINING THREE UNKNOWN NUMBERS.

130. If we have two equations containing *three* unknown numbers, such as

$$2x+3y-5z=9$$
$$x-2y+7z=13,$$

we can eliminate one of these unknown numbers by the methods already explained, giving one equation containing two unknown numbers. Thus, in this particular case, by multiplying the members of the second equation by 2 and subtracting, we would find

$$7y-19z=-17.$$

Since an indefinite number of values will satisfy one equation containing two unknown numbers; therefore, an indefinite number of sets of values will satisfy two equations containing three unknown numbers.

Suppose, however, that we have *three* equations containing three unknown numbers, as

$$2x+3y-5z=9 \quad (1)$$
$$x-2y+7z=13 \quad (2)$$
$$3x-y-2z=8. \quad (3)$$

Eliminating x from (1) and (2) as explained above, we get

$$7y-19z=-17. \quad (4)$$

Multiplying both members of (2) by 3, and subtracting from (3), we obtain

$$5y-23y=-31. \quad (5)$$

Now we can eliminate y from (4) and (5) by multiplying both members of (4) by 5, and both members of (5) by 7, giving

$$35y-95z=-85 \quad (6)$$
$$35y-161z=-217, \quad (7)$$

whence, by subtraction, $66z=132$, (8)

whence $z=2$.

Substituting 2 for z in (4)

$$7y-38=-17, \quad (9)$$

whence $y=3$,

and substituting $y=3$ and $z=2$ in (1), we find $x=5$. Therefore, $x=5$, $y=3$, and $z=2$.

Here we notice that we have been able to find the values of *three* unknown numbers from *three* equations.

131. It is evident that we can proceed in a similar way in any case of three equations containing three unknown numbers. That is, to solve three simultaneous equations containing three unknown numbers :

I. Obtain from two of the equations an equation which contains only two of the unknown numbers, by any method of elimination.

II. From the third given equation and either of the former two, obtain another equation which contains the same two unknown numbers.

III. From the two equations containing two unknown numbers thus found, find the values of these unknown numbers.

IV. By substituting these values in one of the given equations, the value of the remaining unknown number may be found.

132. We will further illustrate this subject by working a few examples. It should be observed that while it makes no difference which one of the unknown numbers we eliminate first, yet the work is often lessened by the selection for this purpose of that one of the unknown numbers whose numerical coefficients have the smallest L. C. M.

(1) Solve
$$4x-5y+z=6. \quad (1)$$
$$7x-11y+2z=9. \quad (2)$$
$$x+y+3z=12. \quad (3)$$

The unknown number z has the smallest numerical coefficients, and it is easier to eliminate it than any of the other unknown numbers. Multiplying both members of (1) by 2, we have

$$8x-10y+2z=12. \quad (4)$$

Subtracting (2) from this, $x+y=3.$ (5)

Now multiply both members of (1) by 3, giving

$$12x-15y+3z=18. \quad (6)$$

Subtracting (3) from this, we find

$$11x-16y=6. \quad (7)$$

We have now to find the values of x and y from (5) and (7). Multiply both members of (5) by 11, giving

$$11x+11y=33. \quad (8)$$

Subtracting (7) from this, we find

$$27y=27, \quad (9)$$

whence $y=1.$

From (5) $x=2$

and from (3) $2+1+3z=12$, whence $z=3.$

Therefore $x=2$, $y=1$, and $z=3.$

The student may verify these in the original equations.

(2) Solve
$$x+y=5. \quad (1)$$
$$y+z=7. \quad (2)$$
$$x+z=6. \quad (3)$$

This is quickly solved by special expedient. Thus add the members of the three equations together, giving

$$2x+2y+2z=18$$

or
$$x+y+z=9. \quad (4)$$

From (1) $x+y=5$, therefore from (4) $z=4.$

From (2) $y+z=7$, therefore from (4) $x=2.$

From (3) $x+z=6$, therefore from (4) $y=3.$

(3) Solve $\frac{6}{x}+\frac{4}{y}+\frac{5}{z}=4.$ (1)

$\frac{3}{x}+\frac{8}{y}+\frac{5}{z}=4.$ (2)

$\frac{9}{x}+\frac{12}{y}-\frac{10}{z}=4.$ (3)

Here we should consider $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ as the unknown numbers. Subtracting (1) from twice (2), we get

$$\frac{12}{y}+\frac{5}{z}=4. \quad (4)$$

Subtracting (3) from three times (2), we get

$$\frac{12}{y}+\frac{25}{z}=8. \quad (5)$$

We are now to find $\frac{1}{y}$ and $\frac{1}{z}$ from (4) and (5). Subtracting (4) from (5),

$$\frac{20}{z}=4,$$

whence $z=5.$

From (4) $y=4.$

From (1) $x=3.$

EXERCISE 74.

EXAMPLES.

Solve the following simultaneous equations:

1. $x+y=37.$
   $x+z=25.$
   $y+z=22.$

2. $2x+y=5.$
   $x+3z=16.$
   $5y-z=10.$

3. $x+y+z=30.$
   $8x+4y+2z=50.$
   $27x+9y+3z=64.$

4. $5x+7y-2z=13.$
   $8x+3y+z=17.$
   $x-4y+10z=23.$

5. $5x-3y=1.$
$9y-2z=12.$
$8x+3z=17.$

6. $x+y-z=17.$
$x+z-y=13.$
$y+z-x=7.$

7. $3x+2y+z=23.$
$5x+2y+4z=46.$
$10x+5y+4z=75.$

8. $4x-2y+5z=18.$
$2x+4y-3z=22.$
$6x+7y-z=63.$

9. $x+y+4z=23.$
$3x-y+2z=11.$
$x+4y-z=4.$

10. $\frac{x}{2}+\frac{y}{3}=12.$
$\frac{z}{5}-\frac{y}{6}=4.$
$\frac{x}{12}+\frac{z}{7}=6.$

11. $\frac{1}{x+y}=1.$
$\frac{2}{x+z}=1.$
$\frac{3}{y+z}=1.$

12. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62.$
$\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47.$
$\frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.$

13. $258-\frac{x}{5}=\frac{y}{7}+\frac{z}{9}.$
$304-\frac{z}{5}=\frac{x}{7}+\frac{y}{9}.$
$296-\frac{y}{5}=\frac{z}{7}+\frac{x}{9}.$

14. $\frac{1}{x}+\frac{1}{y}=1.$
$\frac{1}{x}+\frac{1}{z}=2.$
$\frac{1}{y}+\frac{1}{z}=\frac{3}{2}.$

15. $\frac{2}{x}+\frac{1}{y}=\frac{3}{z}.$
$\frac{3}{z}-\frac{2}{y}=2.$
$\frac{1}{x}+\frac{1}{z}=\frac{4}{3}.$

16. $\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=1.$
$\frac{5}{x}+\frac{4}{y}+\frac{6}{z}=24.$
$\frac{7}{x}-\frac{8}{y}+\frac{9}{z}=14.$

17. $\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=\frac{38}{5}.$
$\frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=\frac{61}{6}.$
$\frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=\frac{161}{10}.$

EXERCISE 75.

LITERAL SIMULTANEOUS EQUATIONS.

Solve the following:

1. $x+y=2a$ and $x-y=2b$.

By addition, $2x=2a+2b$,

whence $x=a+b$.

By subtraction, $2y=2a-2b$,

whence $y=a-b$.

2. $x+y=3a-2b$.
$x-y=2a-3b$.

3. $2x-3y=5b-a$.
$3x-2y=a+5b$.

4. $a(3a+x)=b(b+y)$.
$ax+2by=d$.

5. $ax+by=c$.
$nx+ry=s$.

6. $abx+cdy=2$.
$ax-cy=\frac{d-b}{bd}$.

7. $\frac{1}{x}+\frac{1}{y}=n$.
$\frac{1}{x}-\frac{1}{y}=r$.

8. $\frac{dy}{bx}=\frac{c}{a}$.
$bx+dy=a+c$.

9. $\frac{n}{x}+\frac{1}{y}=a$.
$\frac{r}{x}+\frac{1}{y}=b$.

10. $x+y=2a$.
$x+z=2b$.
$y+z=2c$.

11. $ax+y=r$.
$z-y=s$.
$bx+z=t$.

12. $x+y+z=a$.
$x-y+z=b$.
$x+y-z=c$.

13. $ax+by-cz=2ab$.
$by+cz-ax=2bc$.
$cz+ax-by=2ac$.

14. $x-nx+y=0$.
$y-ry+z=0$.
$x+z=t$.

15. $x+ay+a^2z+a^3=0$.
$x+by+b^2z+b^3=0$.
$x+cy+c^2z+c^3=0$.

16. $x+y+z=a.$

$$\frac{x}{y}=\frac{n}{r}.$$

$$\frac{y}{z}=\frac{p}{q}.$$

17. $\frac{1}{x}=a-\frac{1}{y}.$

$$\frac{1}{y}=b-\frac{1}{z}.$$

$$\frac{1}{z}=c-\frac{1}{x}.$$

18. $\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=n.$

$$\frac{a}{x}-\frac{b}{y}+\frac{c}{z}=r.$$

$$-\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=t.$$

19. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3.$

$$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=1.$$

$$\frac{2a}{x}-\frac{b}{y}-\frac{c}{z}=0.$$

EXERCISE 76.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS.

1. The sum of two numbers is 70, and their difference is 24. Find the numbers.

Problems exactly similar to this have been worked with the use of one unknown number. We will now work it using two unknown numbers.

Let x=the first number,

and let y=the second number.

Then, because the sum of the numbers is 70, and the difference of the numbers is 24; therefore,

$$x+y=70,$$

and

$$x-y=24.$$

Solving these, we find $x=47$ and $y=23$.

2. Find a fraction such that if we add 1 to the numerator, it becomes equal to $\frac{1}{2}$, but if we add 2 to the denominator, it becomes equal to $\frac{1}{3}$.

Let x=the numerator of the required fraction,

and let y=the denominator of the required fraction.

Because the fraction with 1 added to the numerator equals $\frac{1}{2}$, therefore,

$$\frac{x+1}{y}=\frac{1}{2}. \quad (1)$$

Because the fraction with 2 added to the denominator equals $\frac{1}{3}$, therefore,

$$\frac{x}{y+2}=\frac{1}{3}. \tag{2}$$

From (1) and (2) we get, by clearing of fractions,

$$2x+2=y \tag{3}$$

$$3x=y+2. \tag{4}$$

Eliminating y we get $x=4$, whence from (3) $y=10$. The fraction is, therefore, $\frac{4}{10}$.

3. Says A to B: "Give me \$49 and I will have just as much money as you." Says B to A: "Give me \$49 and I will have 3 times as much money as you." How much has each?

Let x=the number of dollars A has.

Then y=the number of dollars B has.

If B gives A \$49, A would have $x+49$ dollars and B would have $y-49$ dollars. Because they would then have equal amounts, therefore,

$$x+49=y-49. \tag{1}$$

If A gives B \$49, B would have $y+49$ dollars and A would have $x-49$ dollars. Because B would then have 3 times as much as A, therefore,

$$3(x-49)=y+49. \tag{2}$$

Eliminating y from (1) and (2), we get

$$2x-196=98, \tag{3}$$

whence

$$x=147. \tag{4}$$

From (1) we then find $y=245$.

Therefore, A has \$147 and B has \$245.

4. A man bought two kinds of cloth, 7 yards of the first kind and 11 yards of the second kind, and paid \$47 for both. If he had bought 11 yards of the first kind and 7 yards of the second kind he would have saved \$4. What was the price per yard of each kind of cloth?

5. Find a fraction such that when 1 is added to both numerator and denominator, it equals $\frac{1}{2}$, but when 3 is subtracted from numerator and denominator it equals $\frac{1}{3}$.

6. Find a fraction such that when 11 is taken from both numerator and denominator it equals $\frac{1}{6}$, but when 12 is taken from both numerator and denominator, it equals $\frac{1}{7}$.

7. A fraction is such that if I add 1 to the numerator, the fraction equals unity; but if I double the first fraction and add 1 to the denominator, the value of the fraction equals unity. What is this fraction?

8. A number is formed of two digits of which the difference is 3. If the digits be reversed, a number is obtained which is $\frac{4}{7}$ of the original number? What is the original number?

Let x represent the digit in tens' place, and y represent the digit in units' place.

9. A farmer sold to one person 9 horses and 7 cows for $1500, and to another, at the same price, 6 horses and 13 cows for the same money. What was the price of each?

10. The sum of two digits which form a number is 9. If we subtract 3 from each digit, the result is 6 more than half the original number. What is the original number?

11. Two masons, A and B, are building a wall, which they could finish, working together, in 12 days. A works 3 days and B 2 days, when the wall is $\frac{1}{5}$ done. How long would it have taken each to have built the wall?

Let x=number of days it would take A,

and y=number of days it would take B.

In one day A builds $\frac{1}{x}$ of the wall, and B $\frac{1}{y}$. But together they build $\frac{1}{12}$ of the wall in one day. Therefore,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{12}. \qquad (1)$$

In 3 days A builds $\frac{3}{x}$ of the wall, and in 2 days B builds $\frac{2}{y}$ of the wall. Therefore, from the problem,

$$\frac{3}{x}+\frac{2}{y}=\frac{1}{5}. \qquad (2)$$

Solving the simultaneous equations (1) and (2), we can find the values of x and y.

12. A and B can together do a certain work in 30 days; at the end of 18 days, however, B is called off and A finishes it alone in 20 days more. Find the time in which each could do the work alone.

13. A cistern holding 4500 gallons is filled by two pipes. If the first pipe be opened 3 minutes and the second pipe 1 minute, 400 gallons will run into the cistern; but if the first pipe be opened 1 minute and the second 7 minutes, 600 gallons will run in. How much water does each pipe carry in one minute? How long will it take both pipes to fill the cistern if they are opened together?

14. A cistern can be filled by two pipes. If both pipes be opened for 15 minutes they will fill $\frac{1}{6}$ of the cistern; but if the first pipe be opened for 12 minutes and the second for 20 minutes, $\frac{1}{5}$ of the cistern will be filled. How long will it take each of the pipes to fill the cistern when opened alone?

15. A man receives \$2160 yearly interest on his capital. If he had loaned the same capital at $\frac{1}{2}$ per cent. higher interest he would receive \$240 more interest each year. Find the amount of his capital and the rate per cent.

16. A man has two sums of money at interest, the first at 4 and the second at 5 per cent. Together they bring in \$3000 interest annually. What is the amount of money loaned at each rate?

17. A man has two sums, one of \$10000 and another of \$15000, at interest, and receives therefrom \$1200 annually. If the first sum had been loaned at the rate that the second bore and if the second sum had been loaned at the rate that the first bore, he would have received \$25 less per year. At what rates were the sums loaned?

18. A and B can do a piece of work in 12 days; B and C can do it in 20 days; A and C in 15 days. How long will it take each to do the work alone?

19. A cistern is filled with three pipes. The first and second will fill it in 72 minutes, the second and third in 120 minutes, and the first and third in 90 minutes. How long would it take each one of the pipes to fill it?

20. Three cities, A, B, and C, are at the corners of a triangle. From A through B to C is 82 miles; from B through C to A is 97 miles; from C through A to B is 89 miles. How far are the cities A, B, and C from one another?

21. A certain number consists of three digits, whose sum is 15. If the first two digits be reversed the number becomes 180 larger, but if the last two digits be reversed, the number becomes but 18 larger. What is the number?

22. The sum of three numbers is 70. The second divided by the first gives 2 for the quotient and 1 for the remainder, but the third divided by the second gives 3 for the quotient and 3 for the remainder. What are the numbers?

23. There was a family of boys and girls and when they were asked how many there were in the family one of the boys said: "I have just as many brothers as sisters." But one of the girls said: "I have twice as many brothers as sisters." How many boys and girls in the family?

24. It takes 72 English and 51 German yards together to make 100 meters. Also 48 English and 84 German yards make 100 meters. How many inches (English) in the meter? How many inches (English) in the German yard?

CHAPTER XII.

POWERS AND ROOTS

EXERCISE 77.

POWERS OF MONOMIALS.

133. The process of raising a number or expression to any power is called **Involution.** We have already learned that $a^n a^r = a^{n+r}$, (1)

where n and r are any positive whole numbers, but where a may be either a whole number or a fraction, and either positive or negative.

Multiply both numbers of equation (1) by a^s and we obtain $a^n a^r a^s = a^{n+r} a^s$. (2)

But the right side of equation (2) is the product of two powers of the same letter, and hence from what we have learned before, the right side of (2) equals a^{n+r+s}.

Hence, $a^n a^r a^s = a^{n+r+s}$. (3)

1. In the same way show that

$$a^n a^r a^s a^t = a^{n+r+s+t}.$$

2. Can you find in the same way the product of more than four factors, each one of which is a power of a?

3. Can you find the product of any number of factors, each one of which is a power of the same number?

4. State then what the product of *any number* of powers of the same number is equal to.

5. In formula (1) can the exponents n and r be equal to each other or must they be different?

If you cannot readily answer this question, turn back to page 55 and see how this formula was proved, and then the answer will be clear

6. What does a^2a^2 equal? What then does $(a^2)^2$ equal?

7. What does a^3a^3 equal? What then does $(a^3)^2$ equal?

8. What does a^4a^4 equal? What then does $(a^4)^2$ equal?

9. What does a^5a^5 equal? What then does $(a^5)^2$ equal?

10. What does a^na^n equal? What then does $(a^n)^2$ equal?

11. In formula (3) can the exponents n, r and s be equal to one another?

12. What does $a^2a^2a^2$ equal? What then does $(a^2)^3$ equal?

13. What does $a^3a^3a^3$ equal? What then does $(a^3)^3$ equal?

14. What does $a^4a^4a^4$ equal? What then does $(a^4)^3$ equal?

15. What does $a^5a^5a^5$ equal? What then does $(a^5)^3$ equal?

16. What does $a^na^na^n$ equal? What then does $(a^n)^3$ equal?

17. What does $a^na^na^na^n$ equal? What then does $(a^n)^4$ equal?

18. What does $a^na^na^na^na^n$ equal? What then does $(a^n)^5$ equal?

19. What does the product of r factors each of which is a^n equal?

20. What then does $(a^n)^r$ equal?

134. These questions lead to the general formula,

$$(a^n)^r = a^{nr}.$$

The truth expressed by this formula may be expressed in words thus: *The r th power of the n th power of a number is equal to the nr th power of that number.*

135. Let us now seek a power of the product of different numbers.

Consider, first, the square of the product of several numbers.

$$(ab)^2 = abab = aabb = a^2b^2;$$

also $$(abc)^2 = abcabc = aabbcc = a^2b^2c^2.$$

21. In the same way show that

$$(abcd)^2 = a^2b^2c^2d^2;$$

also that $$(abcde)^2 = a^2b^2c^2d^2e^2.$$

22. Would a similar result hold if there were more than five factors in the product to be squared?

23. Justify the following statement:

The square of the product of any number of numbers is equal to the product of the squares of those numbers.

Consider, next, the cube of the product of several factors.

$$(ab)^3 = ababab = aaabbb = a^3b^3;$$

also $$(abc)^3 = abcabcabc = aaabbbccc = a^3b^3c^3.$$

24. In the same way show that

$$(abcd)^3 = a^3b^3c^3d^3;$$

also that $$(abcde)^3 = a^3b^3c^3d^3e^3.$$

25. Would a similar result hold if there were more than five factors in the product to be raised to the third power?

26. Justify the following statement:

The cube of the product of any number of numbers is equal to the product of the cubes of those numbers.

27. The fourth power of the product of any number of numbers is equal to what?

28. In the nth power of the product of any number of numbers the first number appears how many times as a factor? The second number appears how many times as a factor? Any one of the numbers appears how many times as a factor? Therefore, the nth power of the product of any number of numbers equals what?

136. Let us now seek a power of the quotient of two numbers.

$$\left(\frac{a}{b}\right)^2 = \frac{a}{b}\,\frac{a}{b} = \frac{a\,a}{b\,b} = \frac{a^2}{b^2};$$

also $$\left(\frac{a}{b}\right)^3 = \frac{a}{b}\,\frac{a}{b}\,\frac{a}{b} = \frac{a\,a\,a}{b\,b\,b} = \frac{a^3}{b^3}.$$

29. In the same way show that

$$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4};$$

also that $$\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5};$$

also that $$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

137. Thus we may raise to any desired power,

First, a power of a number;

Second, the product of several numbers;

Third, the quotient of two numbers.

These three cases include every monomial that can be proposed. Hence any monomial can be raised to any required power by the methods already given.

138. As any *even* power of a monomial having a $-$ sign is the product of an *even* number of subtractive terms, it follows that any *even* power of a monomial having a $-$ sign is a monomial having a $+$ sign or no sign. Thus, $(-2)^2 = +4$, $(-3)^4 = 81$, $(-2ab^2)^6 = 64a^6b^{12}$, etc.

139. Again, as any *odd* power of a monomial having a — sign is the product of an *odd* number of subtractive terms, it follows that any *odd* power of a monomial having a — sign is a monomial having a — sign. Thus, $(-2)^3=-8$, $(-3)^5,=-243$, $(-2ab^2)^7=-128a^7b^{14}$.

Write down the value of each of the following expressions:

1. $(3ab^2)^2$.
2. $(3x^2y)^2$.
3. $(2a^2b^3)^2$.
4. $(-3ax^2)^2$.
5. $(-2c^3d^2)^2$.
6. $(5ab)^3$.
7. $(3a^2x^2)^2$.
8. $\left(\frac{4ab}{3c}\right)^2$.
9. $\left(\frac{5xz}{7y}\right)^2$.
10. $\left(\frac{2x^2z}{5mr^2}\right)^3$.
11. $\left(\frac{4x^2y}{2uv}\right)^3$.
12. $\left(\frac{4rs^2}{6x^2y}\right)^4$.
13. $\left(\frac{6a^2b^2}{9ax}\right)^4$.
14. $\left(\frac{-3x^2y}{-4}\right)^3$.
15. $\left(\frac{5x^4y^2}{2uv}\right)^3$.
16. $\left(-\frac{5x^4y}{2uv^2}\right)^3$.
17. $\left(\frac{rs^2t^3}{-u^3v^2w}\right)^4$.
18. $\left(\frac{x^2yz}{2xy^2}\right)^4$.
19. $\left(\frac{-1}{3x^2z}\right)^4$.
20. $\left(-\frac{4a^2b}{u^2vw^3}\right)^4$.
21. $\left(\frac{ax}{y^2}\right)^3\left(\frac{ax}{y^2}\right)^4$
22. $\left(\frac{uv}{w^2}\right)^4\left(\frac{w}{u^2v}\right)^3$.
23. $\left(\frac{abc^2}{x^2yz}\right)^5\left(\frac{2ax}{byz}\right)^4$.
24. $\left(\frac{2ab}{3xy}\right)^4\left(\frac{-1}{2ax^2}\right)^3$.
25. $\left(\frac{5ab}{2xz}\right)^2\div\left(\frac{2xz}{3az}\right)^3$.
26. $\left(\frac{rst^2}{xy^3}\right)^4\div\left(-\frac{2x}{ry^2}\right)^3$.
27. $\left(\frac{1}{2x^2y}\right)^3\div\left(\frac{1}{2xy^2}\right)^3$.
28. $(3xy^2)^3\div\left(\frac{1}{3xy^2}\right)^3$.
29. $\left(\frac{1}{2ab^3}\right)^3\div(2ab^3)^3$.
30. $\left(\frac{abc}{xyz}\right)^4\div\left(\frac{xyz}{abc}\right)^4$.

EXERCISE 78.

SQUARE OF A BINOMIAL.

140. By actually multiplying $a+b$ by $a+b$ we find that $(a+b)^2=a^2+2ab+b^2$.

As a and b may stand for any numbers whatever, we may say that

The square of the sum of any two numbers is equal to the square of the first number, plus twice the product of the first and second, plus the square of the second.

141. Again by actually multiplying $a-b$ by $a-b$ we find that $(a-b)^2=a^2-2ab+b^2$.

As a and b may stand for any numbers whatever, we may say that

The square of the difference of any two numbers is equal to the square of the first number, minus twice the product of the first and second, plus the square of the second.

The two statements in italics are so important, and are used so often, that they should be thoroughly familiar to the student.

According to the two statements in italics, write down the square of each of the following binomials:

1. $x+y$.
2. $2x+3$.
3. x^2+ab.
4. $uv+w$.
5. $rx+t^2$.
6. $2xy-z^2$.
7. $3x-2y^2$.
8. $4ab+2a^2$.
9. $5x-\frac{y}{5}$.
10. $\frac{u}{v}+w$.
11. $w-\frac{u}{v}$.
12. $4xy-\frac{2a}{b}$.
13. $\frac{x}{y}+\frac{y}{x}$.
14. $\frac{1}{2}+\frac{1}{3}$.
15. $\frac{3}{2}+\frac{1}{3}xy$.
16. $20+5$.
17. $100-1$.
18. $30+7$.

19. $40-3$.

20. x^4-3y^2.

21. $10x+y$.

22. $\frac{x}{2a}-\frac{a}{x}$.

23. $ax^2+\left(\frac{u}{v^2x}\right)^2$.

24. $(3ab^2)^2+2a^2b$.

25. $(5st)^2-\left(\frac{1}{5st}\right)^2$

26. $\frac{2xyz}{3rst}+\frac{2rst}{3xyz}$.

27. $4x^2-\frac{1}{4x^2}$.

28. $(2x)^2+\left(\frac{1}{2x}\right)^2$

29. $(3x^2)^3+(5x)^2$.

30. $2x^2+y^2z^3$.

EXERCISE 79.

CUBE OF A BINOMIAL.

142. In the last exercise we found that

$$(a+b)^2=a^2+2ab+b^2.$$

Multiply each member of this equation by $a+b$, and we obtain

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

The student should actually go through the work of multiplying the second member of the first equation by $a+b$ to see that this result is correct.

As a and b may stand for any numbers whatever, we may say that

The cube of the sum of any two numbers is equal to the cube of the first, plus 3 *times the square of the first multiplied by the second, plus* 3 *times the first multiplied by the square of the second, plus the cube of the second.*

143. In the last exercise we found that

$$(a-b)^2=a^2-2ab+b^2.$$

Multiply each member of this equation by $a-b$, and we obtain

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3.$$

As a and b may stand for any numbers whatever, we may say that

The cube of the difference of two numbers is equal to the cube of the first, minus 3 *times the square of the first multiplied by the second, plus* 3 *times the first multiplied by the square of the second, minus the cube of the second.*

The two statements in italics are so important and so frequently used, that they should be thoroughly familiar to the student.

By means of the two statements in italics write down the cube of each of the following binomials:

1. $x+2y$.
2. $x-2y$.
3. $ab+c$.
4. $c-ab$.
5. $x-y$.
6. $2ax+b$.
7. $2ax-2b$.
8. $3x^2+2x$.
9. $2y^2+3z^2$.
10. $4x-3ab$.
11. $5ab-3ac$.
12. $10+5$.
13. $100-1$.
14. $30+7$.
15. $40-3$.
16. x^4+a^2y.
17. $m^2-r^2s^2$.
18. $(ab^2)^2+(2x)^2$.
19. $(2x)^2+2x^2$.
20. $a+a$.
21. $1-(x^2y^2z^2)^2$.

---

22. $3x+\frac{x}{3}$.
23. $\frac{x}{y}+\frac{a}{b}$.
24. $\frac{2x^2}{3y}+\frac{2y^2}{3x}$.
25. $\frac{1}{a}+\frac{1}{b}$.
26. $\frac{1}{ab}-\frac{1}{ac}$.
27. $\frac{a}{b}\cdot\frac{b}{a}$.
28. $uv+\frac{w}{uv^2}$.
29. $10x-\frac{y}{10}$.
30. $\frac{4n}{3s}-\frac{3s}{4n}$.
31. $\frac{n^2x^2}{2x}-\frac{n^2x^2}{2n}$.
32. $\frac{abc}{rst}+1$.
33. $\frac{3x}{y}-\frac{2yz^3}{x}$.
34. $\frac{x}{a}+\frac{y}{a}$.
35. $ax^2+\left(\frac{u}{v^2x}\right)^2$.
36. $4x^2+\frac{1}{4x^2}$.
37. $4-\frac{1}{4x^2}$.
38. $2a^2-\frac{2}{a}$.
39. $\frac{a^2}{2}-\frac{2^2}{a}$.

EXERCISE 80.

SQUARE OF A POLYNOMIAL.

144. It is so often necessary to square a polynomial that it is well to have a way of writing down the result without being obliged every time to go through all the work of multiplying.

145. Let us first take a polynomial of four terms and find its square by actual multiplication.

$$
\begin{array}{l}
a+b+c+d \\
a+b+c+d \\
\hline
a^2+\ ab+\ ac+\ ad \\
\qquad ab \qquad\quad +b^2+\ bc+\ bd \\
\qquad\quad ac \qquad\quad +\ bc \qquad +c^2+\ cd \\
\qquad\quad\ ad \qquad\quad +\ bd \qquad +\ cd+d^2 \\
\hline
a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2
\end{array}
$$

Slightly changing the order of the terms in this result, we may write

$$(a+b+c+d)^2$$
$$=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

146. The method here used applies to any polynomial, no matter how many terms there may be.

Suppose the polynomial written down twice, the second time immediately below the first, and call the first line the multiplicand and the second line the multiplier, and let us see what terms make up the product, which of course is the square of the given polynomial.

First, it is plain that the square of each term of the polynomial must be part of the product required.

Second, each term of the multiplicand being multiplied by each term of the multiplier, it will be evident, if looked

at carefully, that another part of the product is made up of *twice* the product of each term of the polynomial by each term that *follows* it.

For, consider how any one of these latter described terms can arise. Consider, for example, the term bd in the case worked out of a polynomial of four terms. The only way a term bd can occur in the product is, first, for d of the multiplicand to be multiplied by b of the multiplier, and, second, for b of the multiplicand to be multiplied by d of the multiplier, and the two terms thus produced being combined together we obtain $2bd$ as one term of the product required. Notice that d is one of the letters that *follow* b in the polynomial considered.

The reason that we have no term db in the product is because that term has already been accounted for by considering it the same as bd.

Of course, what is here said about bd will apply to *any* two different terms of the given polynomial.

147. Thus, the square of any polynomial may be written down directly by writing first the *sum of the squares of each of the terms of the given polynomial, and to this sum adding twice the product of each term by each term that follows it in the given polynomial.*

148. In applying this method it must be remembered that the letters we have used may stand for negative as well as positive numbers, and in those terms which are made up of twice the product of each term by each term that follows it, the *rule of signs* must be observed.

By the method just explained the square of $a-x-z$ would be $a^2+x^2+z^2-2ax-2az+2xz$,
the last term having the sign + because the two terms multiplied together to produce this term have *like* signs.

Write down the square of each of the following polynomials:

1. $a+b-c$.
2. $r+s-x$.
3. $m+r-x+y$.
4. $a+2b-c-d$.
5. $a-2b-3c+d$.
6. $2a-2n+3x-y$.
7. $x+y-2u-3v$.
8. x^2+x+1.
9. x^2+2x+1.
10. x^3-3x^2+3x-1.
11. $x^3+3x^2y+3xy^2+y^3$.
12. $x^2+y^2-z^2$.
13. $a^3-x^3-y^3$.
14. r^2-ar+s^2-bs.
15. $a+b+c-d-e$.
16. $2a-2b-c-d-e$.
17. $x+y+z-a-b-c$.
18. $u+2v+3w-4x$.
19. $u^2+2v^2+w+x^2$.
20. $u^2+(2v)^2+(3w)^2-x$.
21. $m+2r-bs+t$.
22. $xy+ab-c^2+z^2$.
23. $abc-xyz+x^2+y^2$.
24. $mrs^2-uv+st-w$.
25. $(ab)^2-x^2y+(2x)^2+y$.
26. $a+2x+ax+2ax^2$.
27. $m+s+xyz-u^2v$.
28. $abx^2-uvy^2+(xz)^2$.

EXERCISE 81.

Roots of Monomials.

149. We have learned how to find any power of a monomial, the square and cube of a binomial, and the square of any polynomial. The reverse process of going back to the number or expression, when the power is given, is called **Evolution**.

150. The number or expression found by evolution is called a **Root** of the number or expression given.

Note that this is an *entirely different* use of the word *root* from that of Art. 103.

151. As there are square, cube, 4th, 5th, etc., powers, so there are square, cube, 4th, 5th, etc., roots.

The *square* root of a given expression means that expression which *squared* will produce the given expression.

The *cube* root of a given expression means that expression which *cubed* will produce the given expression.

The 4th root of a given expression means that expression which raised to the *fourth* power will produce the given expression, etc.

For example, $2^3=8$; therefore, the cube root of 8 is 2, $2^4=16$; therefore, the fourth root of 16 is 2, etc.

152. A root is indicated by the sign $\sqrt{\ }$, called a **Radical Sign.** A horizontal line usually extends from the upper end of the radical sign over the expression of which the root is to be extracted. See Art. 80.

To indicate *what* root is to be extracted, a small figure, called the **Index** of the root, is placed in the angle of the radical, excepting in case of the *square* root, in which the index is almost never used.

Thus, the square root of 16 is indicated by $\sqrt{16}$, the cube root of 8 is indicated by $\sqrt[3]{8}$, the fourth root of 16 is indicated by $\sqrt[4]{16}$.

A letter may be used as the index of a root. Thus, $\sqrt[n]{a}$ means the nth root of a, that is a number which raised to the nth power will produce a.

153. We must notice one important distinction between raising to a power and extracting a root. If we have an expression given to be raised to a given power, we obtain only *one* result, but if we have an expression given to extract a given root we *may* sometimes obtain more than one result.

For example, $5^2=25$, hence we say $\sqrt{25}=5$; but also $(-5)^2=25$; hence we say $\sqrt{25}=-5$.

It appears thus that there are two numbers $+5$ and -5, either of which is a square root of 25. The two results are often written together by means of the double sign $\pm$. Thus, $\sqrt{25} = \pm 5$.

1. What are the *two* square roots of 100?

2. What does 2^4 equal?

3. What does $(-2)^4$ equal?

4. Are there *two* 4th roots of 16?

5. Are there two 4th roots of 81?

6. If n stands for an *even* number is there more than one n th root of a given number? Why?

7. A sixth root of 64 is 2, give another number that is also a 6th root of 64.

8. Any *even* root of a positive number may be either positive or negative. Why?

154. If any number be raised to any *even* power, that number is used an even number of times as a factor, and therefore the result *must* be positive whether the number given was positive or negative. In other words, there is *no* positive or negative number which raised to an *even* power will give a *negative* result, so *we cannot find an even root of a negative number.*

155. If an expression be raised to an *odd* power, that number is used an odd number of times as a factor, and therefore the result is an expression of the *same* sign as the one given. Therefore, *any odd root of an expression has the same sign as the expression itself.*

156. To find any root of any expression we naturally look to see how the corresponding *power* was obtained, and then go through the work backward if possible, thus returning to the expression from which we started in the case of involution.

We have found that $(a^n)^r = a^{nr}$ that is, a^n is an expression which raised to the r th power gives a^{nr}; therefore,

$$\sqrt[r]{a^{nr}} = a^n.$$

Hence, to extract the n th root of a power of an expression, we divide the exponent of the given power by the index of the root, but in order to perform the division, the exponent of the power must be a multiple of the index of the root.

We cannot extract the square root of a^5, because 5, the exponent of the power, is not a multiple of 2, the index of the root.

157. To find the n th root of the product of two factors.

We know that $a^n b^n = (ab)^n$;

therefore, $\sqrt[n]{a^n b^n} = \sqrt[n]{(ab)^n} = ab.$

In this result the first factor, a, may be found by taking the n th root of a^n, the first factor of the given expression, and the second factor, b, of the result may be found by taking the n th root of b^n, the second factor of the given expression. Therefore, the n th root of the product of two factors is found by taking the product of the n th roots of those factors.

158. Of course the same argument may be used with more than two factors, and hence, evidently, *the n th root of the product of several factors is found by taking the product of the n th roots of those factors.*

159. To find the nth root of the quotient of two expressions.

We know that $$\frac{a^n}{b^n}=\left(\frac{a}{b}\right)^n$$

therefore $$\sqrt[n]{\frac{a^n}{b^n}}=\sqrt[n]{\left(\frac{a}{b}\right)^n}=\frac{a}{b}$$

In this result the numerator, a, is found by taking the nth root of the given numerator, and the denominator, b, is found by taking the nth root of the given denominator. Therefore, *the nth root of the quotient of two expressions is found by taking the quotient of the nth roots of those expressions.*

EXERCISE 82.

EXAMPLES.

Find the square root of each of the following twelve expressions:

1. a^2x^4.
2. $9a^6x^2$.
3. $\frac{9a^2}{16x^4}$.
4. $r^2s^2t^4$.
5. $\frac{r^2x^2}{a^2y^2}$.
6. $\frac{a^4b^2}{49a^2x^2}$.
7. $\frac{49x^2y^4}{36x^4y^2}$.
8. $\frac{x^2z^4}{y^4z^8}$.
9. $\frac{4}{4x^6y^8}$.
10. $\frac{x^2}{y^4}\div\frac{u^4}{v^2}$.
11. $\frac{a^2}{b^4}\,\frac{c^6}{d^8}$.
12. $m^2r^4\div\frac{a^4}{b^6}$.

Find the cube root of each of the following ten expressions:

13. a^3x^6.
14. $8a^9b^6c^3$.
15. $\frac{x^6y^9z^{15}}{27}$.
16. $-64a^9x^{12}$.
17. $\frac{n^6x^{15}}{-8a^3}$.
18. $\frac{-8a^3}{27x^6}\,\frac{m^6}{t^9}$.

19. $\frac{27}{-8x^6}\,\frac{x^3}{64}$.

20. $-64a^6b^9 \div 8a^3$.

21. $\frac{x^6y^6z^6}{u^3v^3w^3} \div \frac{-8}{x^3y^6}$.

22. $-\frac{a^3x^3}{m^3} \div -\frac{r^6x^6}{n^6}$.

23. Find the fourth root of $a^8b^{12}c^4$.

24. Find the square root of $16a^8x^4y^{12}$, and then the square root of this result.

25. Find the fourth root of $16a^8x^4y^{12}$.

26. Find the cube root of $a^{36}b^{12}c^{24}$, and then the fourth root of this result.

27. Find the sixth root of $a^{36}b^{12}c^{24}$, and then the square root of this result.

28. Find the twelfth root of $a^{36}b^{12}c^{24}$.

29. Find the square root of $a^{36}b^{12}c^{24}$, then the cube root of the result, and then the square root of this second result.

30. Find the fifth root of $-32a^5x^{10}y^{10}$.

31. Find the seventh root of $128a^7b^7c^{14}$.

32. Find the square root of $\frac{\sqrt[4]{a^8x^{16}y^8}}{x^2y^4}$.

160. Before leaving the subject of the roots of monomials, it is well to notice that what we have learned may be used to find the roots of arithmetic numbers, when the numbers given have exact roots.

We resolve the number into its prime factors, and express it as the product of various powers of these prime factors, then divide each exponent by the index of the required root. When the resulting factors are multiplied together the required root is found.

Suppose we wish to find the square root of 53361.

3	53361
3	17787
7	5929
7	847
11	121
	11

Hence, $53361 = 3^2 \times 7^2 \times 11^2$;

therefore, $\sqrt{53361} = 3 \times 7 \times 11 = 231$.

33. Find the square root of 5184.
34. Find the square root of 43204.
35. Find the cube root of 85184.
36. Find the cube root of 32768.

EXERCISE 83.

Square Root of Polynomials.

161. To find out how to extract the square root of a polynomial we must see how the polynomial was produced by squaring. We know that

$$(x+y)^2 = x^2 + 2xy + y^2.$$

Therefore, we know that the square root of $x^2+2xy+y^2$ is $x+y$.

Our problem then is this: Given the expression, $x^2+2xy+y^2$, to find from it the expression, $x+y$.

The first term, x, of the root is the square root of the first term, x^2, of the given expression.

Let us set down this term, x, already found, and subtract its square from the given expression. There remains of the given expression $2xy+y^2$ or $(2x+y)y$.

From this we see that the second term, y, of the root will be the quotient when the remainder just found is divided by $2x+y$.

This divisor, $2x+y$, consists of two terms, the first of which is twice the portion of the root already found, and the second is the new term, y, itself.

The work may be arranged as follows:

$$\begin{array}{r|l} & \text{Given Expression.} \quad \text{Root.} \\ & x^2+2xy+y^2 \;(\; x+y \\ & x^2 \\ \hline 2x+y & 2xy+y^2 \\ & 2xy+y^2 \\ \hline \end{array}$$

After the first term x, of the root has been found, its double, $2x$, is used as a *trial* divisor by which to divide the remainder, $2xy+y^2$. We see that the first term, $2xy$, of this remainder when divided by the trial divisor, $2x$, gives y, from which we judge that y is the next term of the root. When the y is thus found it is added to the *trial* divisor, $2x$, giving the complete divisor, $2x+y$, and *this* is multiplied by y, giving the expression, $2xy+y^2$.

162. Of course we may obtain by this process the *difference* of two numbers for our square root as well as the *sum*, as in the example just given. This will be plain by working out another example.

To find the square root of $4a^2-12ab+9b^2$.

Arrange the work thus:

$$\begin{array}{r|l} & 4a^2-12ab+9b^2 \;(\; 2a-3b \\ & 4a^2 \\ \hline 4a-3b & -12ab+9b^2 \\ & -12ab+9b^2 \\ \hline \end{array}$$

Here the first term of the remainder, $-12ab$, when divided by the trial divisor, $4a$, gives the quotient, $-3b$. Hence we judge that $-3b$ is the next term of the root, and upon trial this proves to be right.

Find the square root of each of the following expressions:

1. $a^2+4ab+4b^2$.
2. $4a^2-4ab+b^2$.
3. $9a^2-18ab+9b^2$.
4. $4a^2-16a+16$.
5. $a^4-2a^2b^2+b^4$.
6. $a^2-2ab^3+b^6$.
7. $x^4+2^2y^2+y^4$.
8. x^4+2x^2+1.
9. $x^4+2x^3+x^2$.
10. $x^6-2x^4+x^2$.
11. $x^6-2x^5+x^4$.
12. $9x^4-18x^2+9$.
13. $a^6-2a^3x^3+x^6$.
14. $a^2b^2+2abcd+c^2d^2$.
15. $a^2x^4-2abx^3+b^2x^2$.
16. $4x^6-4nx^3y+n^2y^2$.
17. $9a^4b^2-18a^3b^3+9a^2b^4$.
18. $a^4b^4-6a^2b^2+9$.
19. $a^4b^6-2a^2b^3c^5+c^{10}$.
20. $a^2b^2x^2y^2-8abcx^2y+16c^2x^2$.

163. Thus far the polynomials of which we have extracted the square root have been in every case those of three terms.

The above process, however, can be extended so as to find the square root of any polynomial which is a perfect square, no matter how many terms the polynomial contains. For example, to find the square root of

$$a^2+b^2+c^2+2ab+2ac+2bc.$$

First arrange the expression according to powers of some letter, say a, and write

$$a^2+2ab+2ac+b^2+2bc+c^2.$$

The first term, a^2, of this polynomial is produced by squaring a. Therefore, the first term of the root is a, and the *whole* root is a + something, and this something is what we wish to find.

Proceeding as before with the first term of the root, a part of the process may be arranged thus:

$$\begin{array}{l} a^2+2ab+2ac+b^2+2bc+c^2 \ (\ a \\ a^2 \\ \hline 2ab+2ac+b^2+2bc+c^2 \end{array}$$

Now twice a used as a trial divisor would suggest b for the next term of the root.

Call the next term b and proceed as before, and the work would stand thus:

$$\begin{array}{l|l} & a^2+2ab+2ac+b^2+2bc+c^2 \ (\ a+b \\ & a^2 \\ \hline 2a+b & 2ab+2ac+b^2+2bc+c^2 \\ & 2ab\quad\ +b^2 \\ \hline & 2ac\quad\ +2bc+c^2 \end{array}$$

There is *still* a remainder, so we have not yet found the entire root, but the root is $a+b+$ something, and this something is what we wish to find.

Now let us consider $a+b$, the part of the root already found, as a single term, and use it as we have before used the first term of the root. We must then take twice $a+b$ and use it as a trial divisor by which to divide the last remainder, $2ac+2bc+c^2$, from which we judge that c is the next term of the root. When the c is thus found it is added to the *trial* divisor, $2a+b$, giving the complete divisor, $2a+2b+c$, and *this* is multiplied by c, giving the expression $2ac+2bc+c^2$.

The work from the beginning will now stand thus:

$$\begin{array}{l|l|l} & a^2+2ab+2ac+b^2+2bc+c^2 \ (\ a+b+c \\ & a^2 \\ \hline 2a+b & 2ab+2ac+b^2+2bc+c^2 \\ & 2ab\quad\ +b^2 \\ \hline 2a+2b+c & 2ac\quad\ +2bc+c^2 \\ & 2ac\quad\ +2bc+c^2 \end{array}$$

As there is now no remainder the process is ended and the root is $a+b+c$. If after finding the third term of the root there were still a remainder, we would group the three terms thus found into a single term, and use this group as we have always used the first term of the root.

The process may evidently be extended to finding the square root of any polynomial that is a perfect square, no matter how many terms the polynomial may contain.

164. From what has been said we see that the **Method of finding the Square Root of any Polynomial** is as follows:

I. Arrange the terms according to the powers of some letter.

II. Find the square root of the first term, write it as the first term of the root, and subtract its square from the given expression.

III. Use twice the portion of the root already found as a trial divisor, and divide the remainder just found by this trial divisor. Add the quotient to the root and also to the tria divisor.

IV. Multiply the complete divisor by the term of the root last obtained and subtract the product from the remainder.

V. Repeat III and IV until there is no remainder.

Find the square root of each of the following polynomials:

11. $x^2+4xy+2xz+4y^2+4yz+z^2$.

12. $x^26xy-4xz+9y^2+12yz+4z^2$.

13. $4x^2+16xy-4xz+16y^2-8yz+z^2$.

14. $a^4-2a^2b+2a^2c+b^2-2bc+c^2$.

15. $4a^4-12a^2b^2-8a^2c+9b^4+12b^2c+4c^2$.

16. $a^4+2a^2b^2+2a^2c^2+b^4+2b^2c^2+c^4$.

17. $a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2$.

18. $a^2+4ab-6ac+4b^2+12bc+9c^2$.

19. $a^6+2a^5+3a^4+4a^3+3a^2+2a+1$.

20. $9x^4+18a^2x^2+6x^2+9a^4+6a^2+1$.

21. $\frac{x^4}{a^4}+2\frac{x^3}{a^3}+3\frac{x^2}{a^2}+2\frac{x}{a}+1$.

22. $x^8+2x^6+3x^4+2x^2+1$.

23. $a^4b^4+2a^2b^2cd-4a^2b^2e+c^2d^2-4cde+4e^2$.

24. $a^6b^2+2a^3bc^2d+2a^3be+c^4d^2+2c^2de+e^2$.

25. $a^2b^2c^2+2abcde+2abcf+d^2e^2+2def+f^2$.

EXERCISE 84.

Cube Root of Polynomials.

165. To find how to extract the cube root of a polynomial we must see what the cube of an expression is and then return, if possible, from the cube back to the expression from which this cube was obtained.

166. We know that

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3.$$

Our problem, then, is this: Given the expression, $x^3+3x^2y+3xy^2+y^3$, to find from it the expression, $x+y$, which is the cube root of the given expression.

167. We see that the first term, x, of the root is the cube root of x^3, the first term of the given expression.

Set down the x as the first term of the root and subtract its cube from the given expression, and we have a remainder,

$$3x^2y+3xy^2+y^3, \text{ or } (3x^2+3xy+y^2)y.$$

We see from this that the second term, y, of the root is the quotient obtained by dividing this remainder by $3x^2+3xy+y^2$.

Now this divisor is composed of three terms, of which the first is 3 times the square of the first term of the root.

the second is 3 times the product of the first term of the root and the new term, y, and the third is the square of the new term of the root.

The sum of these three terms constitute the complete divisor, while the remainder found by subtracting x^3 from the given expression is the complete dividend.

This complete divisor contains two terms which involve the y, which is not yet supposed to be known. However, we may get something of an idea of what the second term of the root must be by using the *first term* of the above remainder as a *trial* dividend and 3 times the square of the first term of the root as a *trial* divisor, and then if the number we get by this division is correct the complete divisor (which can then be found) when multiplied by the new term of the root must give the complete dividend, *i. e.*, the remainder.

168. The work may be arranged as follows:

$$\begin{array}{rl} & x^3+3x^2y+3xy^2+y^3 \ (\ x+y \\ & x^3 \\ \hline 3x^2+3xy+y^2 & \left|\begin{array}{l} 3x^2y+3xy^2+y^3 \\ 3x^2y+3xy^2+y^3 \end{array}\right. \end{array}$$

169. The remark made under square root (Art. 162) about the — sign applies here as well, as is illustrated in the following example:

Find the cube root of $x^3-6x^2y+12xy^2+y^3$.

The work is as follows:

$$\begin{array}{rl} & x^3-6x^2y+12xy^2-8y^3 \ (\ x-2y \\ & x^3 \\ \hline 3x^2-6xy+4y^2 & \left|\begin{array}{l} -6x^2y+12xy^2-8y^3 \\ -6x^2y+12xy^2-8y^3 \end{array}\right. \end{array}$$

Find the cube root of the following expressions:

1. $a^3+6a^2b+12ab^2+8b^3$.
2. $8a^3+12a^2b+6ab^2+b^3$.
3. $x^3y^3+12x^2y^4+48xy^5+64y^6$.
4. $8a^3-60a^2x+150ax^2-125x^3$.
5. $27x^6-54x^4yz+36x^2y^2z^2-8y^3z^3$.
6. $8x^6+12x^4+6x^2+1$.
7. $a^3x^6-3a^2bx^4y^2+3ab^2x^2y^2-b^3y^6$.
8. $8a^3b^3-24a^2b^2c+24abc^2-8c^3$.
9. $\frac{x^6}{a^6}+3\frac{x^4}{a^3}+3x^2+a^3$.
10. $\frac{x^3}{a^6}+3\frac{x^2}{a^2b}+3\frac{a^2x}{b^2}+\frac{a^6}{b^3}$.

170. So far all the expressions of which the cube root was required were polynomials of four terms, but we may have a polynomial of more than four terms of which the cube root is required. In this case the process already given may be extended as in the case of the square root, viz.: Find two terms of the root, as already explained, and then consider these two terms as a single term and use their sum the same as a single term was used before.

Find the cube root of each of the following three expressions:

11. $a^6+3a^5+6a^4+7a^3+6a^2+3a+1$.
12. $x^6-6x^5+9x^4+4x^3-9x^2-6x-1$.
13. $64x^6+192x^5+144x^4-32x^3-36x^2+12x-1$

CHAPTER XIII.

HARDER FACTORS, MULTIPLES, AND FRACTIONS.

EXERCISE 85.

FACTORS COMMON TO ALL THE TERMS OF AN EXPRESSION.

171. In Chapter IX the subject of factors, multiples, and fractions was treated to some extent, but the work there was confined to the case of monomials. We now resume the same subject, but treat of more complicated expressions than before.

172. The simplest case of factors of polynomials is where the same factor is seen to be common to all the terms of the polynomial. In this case the polynomial may be written in a simpler form, by dividing each term by this common factor, enclosing the quotient in a parenthesis, and writing the common factor outside the parenthesis as a multiplier.

EXAMPLES.

1. Factor ax^2+ax+a.

Here a is seen to be a common factor of each term; therefore, removing this factor we have

$$ax^2+ax+a=a(x^2+x+1.)$$

2. Factor $5ax^3+15ax^2+20ax+50a$.

Here $5a$ is seen to be a factor of each term; therefore, removing this factor, we have

$$5ax^3+15ax^2+20ax+50a=5a(x^3+3x^2+4x+10.)$$

Find the factors of each of the following expressions:

3. $x^2+x^3+x^4$.

4. $m^2x^2+n^2x^2+r^2x^2$.

5. $a^2bc+a^2bd^2+a^2be$.

6. $rstx^2+rst^2y^2+rs^2t^3z^2$.

7. $auv^3+buvx+uvw$.

8. $mxyz+nx^2y^2z^2+rx^3y^3z^3$.

9. $6a^3+2a^4+4a^5$.

10. $10a^2b^2-15ab^3-25ab^2c$.

11. $3m^2x^2-5m^2y^2-9m^2x^2-15m^2y^2$.

12. $33r^2x^3+55r^2x^4y-66r^2x^2y^3$.

173. Sometimes *one* factor is common to *some* of the terms of an expression, and *another* factor common to *other* terms. In this case you can *sometimes*, though not always, simplify the expression by taking out the common factors where they can be taken out.

13. Factor $ax+ay-az+bx+by-bz$.

Here the first three terms have a common factor a, and the last three terms have a common factor b. Taking out a from the first three terms and b from the last three terms, we may write

$$ax+ay-az+bx+by-bz=a(x+y-z)+b(x+y-z.)$$

Now it is plain that in *this* expression the factor $(x+y-z)$ is common. Hence, we may write

$$a(x+y-z)+b(x+y-z)=(a+b)(x+y-z).$$

Therefore, putting the expression we started with equal to this last, we get $ax+ay-az+bx+by-bz=(a+b)(x+y-z)$.

Factor each of the following expressions:

14. $ax+bx+cx+2ay+2by+2cy$.

15. $ax+2ay+3az-2bx-4by-6bz$.

16. $ax^2y+bx^2z+cx^2u+axy+bxz+cxu$.

17. $x^3y^2-x^3y+x^3+y^2-y+1$.

18. $abxyz+abxuv+abxw+abyz+abuv+abw$.

19. $abxyz^2+abxyz+abxy+abxz^2+abxz+abx$.

20. $auvwx+buvwx+auvwy+buvwy$.

21. $xy+ax+ay+a^2$.

22. $xy-ay-bx+ab$.

23. $x^2y-ay-2b^2x^2+2ab^2$.

EXERCISE 86.

FORMATION OF CERTAIN PRODUCTS.

174. In order readily to factor expressions, we must know what expressions were multiplied together to produce the expression which is given us to factor. This knowledge is gained by practice in writing down the products of a few forms of expressions. Those we take are *binomial* factors. We have already found out in the chapter on multiplication how to multiply two binomials together, but it is very important that the student should be able to write down rapidly certain products *by inspection*, that is, without actually going through the work of multiplication.

175. The first case to consider is that of the product of the sum of two numbers by the difference of those numbers.

Thus, suppose it is required to write out the product $(a+b)(a-b)$. By multiplication we find that

$$(a+b)(a-b)=a^2-b^2.$$

That is, the *difference of the squares* of a and b.

Now as a and b may stand for any numbers whatever, we may state that

The sum of any two numbers multiplied by the difference of those numbers is equal to the difference of the squares of those numbers.

EXAMPLES.

Write down by inspection each of the following products:

1. $(x+a)(x-a)$.
2. $(x+1)(x-1)$.
3. $(x+2y)(x-2y)$.
4. $(m+n)(m-n)$.
5. $(a^2+b)(a^2-b)$.
6. $(x^2+4)(x^2-4)$.

7. $(ax^2-by^2)(ax^2+by^2)$.
8. $(abc+a^2bc^3)(abc-a^2bc^3)$.
9. $(x^2+4)(x^2-4)$.
10. $(x^5-2ab)(x^5+2ab)$.

176. The second case of the multiplication of two binomials is that where the first term is the same in each binomial.

Thus, suppose it required to write out the product $(x+a)(x+b)$. By actual multiplication we could find that

$$(x+a)(x+b)=x^2+ax+bx+ab$$
$$=x^2+(a+b)x+ab.$$

Notice that the first term of the product is x^2, the second term is x with a coefficient equal to the sum of the second terms of the given binomials, and the third term of the product is the product of the second terms of the given binomials.

As x, a, and b are not in any way restricted, but may stand for *any* numbers whatever, the result reached is perfectly general, and therefore

The product of any two binomials in which the first terms are alike, is equal to the square of the first term, plus the first term with a coefficient equal to the sum of the second terms, plus the product of the second terms.

Of course, due attention must be paid to the signs of the terms in writing out such products. Thus,

$$(x-5)(x+2)=x^2+(-5+2)x+(-5\times 2)$$
$$=x^2-3x-10.$$

Write down the products in each of the following:

11. $(x+a)(x+2)$.
12. $(x-2)(x+1)$.
13. $(x+3)(x-5)$.
14. $(x+5)(x+6)$.
15. $(x-5)(x-6)$.
16. $(x+2)(x-15)$.

17. $(x+1)(x+5)$.
18. $(x-4)(x-y)$.
19. $(x+1)(x-5)$.
20. $(x+2)(x+4)$.
21. $(x-1)(x+7)$.
22. $(x+10)(x-11)$.
23. $(x+25)(x-4)$.
24. $(x-a)(x+3)$.
25. $(x-10)(x-10)$.
26. $(x+10)(x-5)$.
27. $(x+5)(x+12)$.
28. $(x-4)(x-15)$.
29. $(x+3)(x+20)$.
30. $(x+2)(x+30)$.
31. $(x-1)(x-5)$.
32. $(x-2)(x-3)$.
33. $(x+2)(x+3)$.
34. $(x-6)(x+5)$.
35. $(x+2a)(x+2b)$.
36. $(x-3a)(x+b^2)$.
37. $(x-a^2)(x+b^2)$.
38. $(x^2-1)(x^2+2)$.
39. $(a+bc)(a-cd)$.
40. $(a-mr^2)(a-ms^2)$.
41. $(x+my)(x+ny)$.
42. $(x^2-a^2y^2)(x^2-ay)$.
43. $(2a-b)(2a+3b)$.
44. $(2a-b^2)(2a-c^2)$.
45. $(x+3yz)(x-2uv)$.
46. $(x^2+a^2)(x^2+1)$.
47. $(x^2+2ay)(x^2-3ay)$.)
48. $(ab^2+c^2)(ab^2-2)$.
49. $(mn+5)(mn-4)$.
50. $(x^2y+xy^2)(x^2y-xy^2)$.

EXERCISE 87.

EXPRESSIONS OF THE FORM x^2-a^2.

177. In this form, x^2-a^2, the letters x and a can of course stand for any two numbers whatever; so to speak of expressions of the form x^2-a^2 is the same as saying, an expression composed of the difference of two square numbers.

178. We have learned how to write down at once the product of the sum of two numbers multiplied by the difference of those numbers, and we now take up the reverse process of finding the two factors when their product is given.

As the sum of two numbers multiplied by the difference of those numbers is equal to the difference of the squares of those numbers, it follows that the difference of two numbers, *each of which is a perfect square*, is equal to the sum of the *square roots* of those numbers multiplied by the difference of the *square roots* of those numbers.

EXAMPLES.

Find two factors of each of the following expressions:

1. x^2-4.
2. x^2-9.
3. x^2-1.
4. x^2-a^2.
5. x^2-4y^2.
6. $a^2-4x^2y^2$.
7. $a^2b^2-25c^2$.
8. $4a^2x^2-y^4$.
9. $9b^4-9a^4$.
10. $16n^2-36r^2x^2$.
11. $x^2-4a^2y^4$.
12. a^8-a^6.
13. $4a^8-16a^6x^8$.
14. $64m^8-81n^6$.
15. $100-25$.
16. $2500-16$.
17. $121-81x^8$.
18. $625-625a^2x^2$.
19. $81x^4-81y^8$.
20. $49a^2x^2-49a^2y^2z^4$.

The following expressions require a factor to be removed from each term before they are in the form of the difference of two squares:

21. $5a^2-5b^2y^4$.
22. $20x^6-20a^6$.
23. $10a-10ar^2x^2$.
24. $a^3x^4-a^5x^2y^6$.
25. $5ax^3y-5ax^7$.
26. $mx^2-mn^2x^4$.
27. $\frac{1}{10}x^3y^2-\frac{1}{10}x^5y^4$.
28. $m^3x^3-m^5x^7y^4$.
29. $2x^5-50xy^8$.
30. $27a^3-27ab^2$.
31. xyz^2-9xy^3.
32. $x^2z^3-4y^4z^3$.
33. $49n^4y-196n^2y^3$.
34. $50n^4y-98n^2y$.
35. $28xy^3-63x^3y$.
36. $9nx^4-36ny^2$.
37. ax^3-ax.
38. $uvw^2-4uv^3w^4$.
39. $24a^2x^6-54b^2y^6$.
40. $50ax^2y-18ax^4y$.

EXERCISE 88.

EXPRESSIONS OF THE FORM x^2+ax+b.

179. We have learned how to write down the product of two binomial factors, when the first terms of the two factors were alike, and found the result was always a *trinomial.* We further saw the relation this trinomial sustained to the two binomial factors which were multiplied together to produce it.

180. We now take up the reverse process of returning from the product to the two factors which were multiplied together to produce it.

We can factor any expression of the form x^2+ax+b if we can find two numbers whose sum is a and whose product is b.

It will assist some in finding the two factors, if the student will remember, that when the third term of the given expression is positive, the second terms of the two factors will have like signs, and when the third term of the given expression is negative, the second terms of the two factors have unlike or opposite signs.

EXAMPLES.

Find the factors of each of the following expressions:

1. x^2+8x+7.

The first term of each binomial factor will, of course, be x, and we are to find the second terms of the binomial factors. To do this we must find two numbers whose sum is 8 and whose product is 7. Obviously 7 and 1 are the only numbers that have 8 for their sum and 7 for their product. Therefore, we conclude that the two factors sought are $x+7$ and $x+1$.

2. x^2-2x-3.

Here we must find two numbers whose sum is -2 and whose product is -3, Obviously -3 and 1 are the only numbers whose sum is -2 and whose product is -3. Therefore the factors are $x-3$ and $x+1$.

3. x^2+5x+6.
4. x^2+5x+4.
5. $x^2+8x+15$.
6. $x^2+11x+28$.
7. $x^2+55x+250$.
8. $x^2+20x+100$.
9. $x^2+10x+21$.
10. $x^2+11x+18$.
11. x^2-9x+8.
12. $x^2-16x+48$.
13. x^2+6x+9.
14. $x^2+8x+16$.
15. x^2+4x+4.
16. $x^2+12x+36$.
17. $x^2+12x+35$.
18. $x^2-10x+16$.
19. x^2+2x-8.
20. $x^2+13x+42$.
21. $x^2-4x-60$.
22. $x^2+22x+120$.
23. $x^2+110x+1000$.
24. $x^2+52x+100$.
25. $x^2-48x-100$.
26. $x^2+15x+36$.

In the above examples the first term is x^2, and the second term is some multiple of x. Of course some other expressions than x^2 and a multiple of x might be used for the first and second terms respectively, provided only that the first term is the square of something, and the second term is a multiple of the square root of the first term, as in the following examples:

27. $a^2x^2-4ax-12$.
28. x^6-2x^3-35.
29. $r^2x^4-16rx^2+60$.
30. $n^4a^4+31n^2a^2+30$.
31. $a^2r^4x^8+17ar^2x^4+16$.
32. m^4+17m^2+30.
33. $n^4x^4+70n^2x^2+600$.
34. $x^2y^2-10xy-200$.
35. a^6-17a^3+70.
36. $4x^4+20\times 2x^2+36$.
37. $4x^2+40x+36$.
38. $a^4b^4-12a^2b^2+11$.
39. $a^2x^4-5ax^2y-50y^2$.
40. $\frac{a^2}{b^2}-20\frac{a}{b}+100$.

EXERCISE 89.

Expressions of the Form a^3-b^3.

181. By trial we find that a^3-b^3 can be divided by $a-b$ as follows:

$$\begin{array}{l} a-b\ )\ a^3-b^3\ (\ a^2+ab+b^2 \\ \qquad\quad \underline{a^3-a^2b} \\ \qquad\qquad\quad a^2b-b^3 \\ \qquad\qquad\quad \underline{a^2b-ab^2} \\ \qquad\qquad\qquad\quad ab^2-b^3 \\ \qquad\qquad\qquad\quad \underline{ab^2-b^3} \end{array}$$

Now remembering that the divisor multiplied by the quotient equals the dividend, we learn from this division that $a^3-b^3=(a-b)(a^2+ab+b^2)$;
and as a and b stand for any numbers whatever, we can make the following statement:

The difference between any two numbers, each of which is a perfect cube, can be expressed as the product of two factors, one of which is the difference between the cube roots of the numbers given, and the other is the square of the cube root of the first number plus the product of the cube roots of the two numbers plus the square of the cube root of the second number.

Examples.

Factor each of the following expressions:

1. a^6-8.

Here we have the difference between two numbers, each of which is a perfect cube, $\sqrt[3]{a^6}=a^2$ and $\sqrt[3]{8}=2$. Hence, we write the two factors $(a^2-2)(a^4+2a^2+4)$.

2. x^3-y^3.
3. $x^3-a^3y^3$.
4. $x^6-n^3y^6$.
5. $a^3b^3-c^3d^3$.
6. $n^6r^9-n^9r^6$.
7. $8x^6-27a^6y^6$.

8. $64x^9-125x^3y^6$.

9. $8a^3x^3-27a^6x^6$.

10. $27a^6x^3-27a^3x^6$.

11. $x^3-8a^3b^6c^9$.

12. x^6y^9-1.

13. $1-a^6x^9$.

14. $125-27$.

15. $1000-64$.

16. $u^3v^3w^3-x^6y^6z^6$.

17. $125-a^3b^3c^6$.

18. $125a^3-64a^6x^6y^6$.

19. $1000x^3-64y^3z^3$.

20. $x^6y^9-u^3v^6w^9$.

21. $8a^3x^6y^9-27a^9x^6y^3$.

22. $\frac{a^3}{b^6}-\frac{64u^3}{v^3}$.

23. $\frac{64x^3y^6}{27a^3}-\frac{u^3v^3}{27b^3}$.

24. $\frac{a^3b^3}{c^3d^3}-\frac{64x^3}{27y^3}$.

25. $\frac{1000}{a^3b^3}-\frac{27x^3}{a^3b^3}$.

Sometimes an expression assumes the form of the difference of two cubes, after a factor has been removed from each term, as in the following examples:

26. $a^2x^3-a^5$.

If we take out the factor a^2 there will remain x^3-a^3 of which the factors are $x-a$ and x^2+ax+a^2. Hence, the factors of the given expression are a^2, $x-a$, and x^2+ax+a^2. Hence the given expression equals $a^2(x-a)(x^2+ax+a^2)$.

27. $x^3z^2-y^3z^2$.

28. $a^3b^6x^2-c^9x^2$.

29. $a^3b^3x^2y-c^6x^2y$.

30. $r^4s^4t^7-27r^7s^4t^4$.

31. $\frac{a^4nv}{27x^3}-au^4v^4$.

32. $3m^5n^2-3m^2n^8$.

33. $5a^3y^3-40b^3x^3$.

34. $2a^2x^6y^6-2a^2$.

35. $2a^2x^6y^3-\frac{2a^2}{a^6y^9}$.

36. $\frac{5a^3}{b^3}-\frac{5x^3}{8y^3}$.

EXERCISE 90.

EXPRESSIONS OF THE FORM a^3+b^3.

182. By actual division we find that a^3+b^3 can be divided by $a+b$ as follows:

$$\begin{array}{l} a+b \;)\; a^3+b^3 \;(\; a^2-ab+b^2 \\ \quad a^3+a^2b \\ \quad \overline{\quad -a^2b+b^3} \\ \quad -a^2b-ab^2 \\ \quad \overline{\qquad\qquad ab^2+b^3} \\ \qquad\qquad ab^2+b^3 \\ \qquad\qquad \overline{\qquad\qquad} \end{array}$$

Now remembering that the divisor multiplied by the quotient equals the dividend we learn from this division that $a^3+b^3=(a+b)(a^2-ab+b^2)$;
and as a and b may stand for any numbers whatever, we may make the following statement:

The sum of any two numbers, each of which is a perfect cube, can be expressed as the product of two factors, one of which is the sum of the cube roots of the numbers given, and the other is the square of the cube root of the first number minus the product of the cube roots of the two numbers plus the square of the cube root of the second number.

EXAMPLES.

Factor each of the following expressions:

1. a^6+8.

Here we have the sum of two numbers, each of which is a perfect cube $\sqrt[3]{a^6}=a^2$ and $\sqrt[3]{8}=2$. Hence we may write

$$a^6+8=(a+2)(a^2-2a+4).$$

2. x^3+y^3.
3. $a^3x^3+b^3y^3$.
4. $m^3+x^3y^3$.
5. $m^6+a^6b^3$.
6. $a^3b^6+8a^6b^3$.
7. $8x^3y^6+27y^3$.
8. $64x^3y^6+125z^6$.
9. $64x^9+8x^9y^6$.

10. m^9+n^9.
11. r^6+x^6.
12. $u^6v^6x^9+u^3v^3x^6$.
13. $x^3y^3z^3+1$.
14. $a^3b^3c^3+x^6y^6z^6$.
15. $8a^3b^3x^3+8a^3b^3y^3$.
16. $x^6y^9+27x^6z^9$.
17. $64x^6+125a^3b^6c^9$.
18. $1000+64$.
19. $64+8$.
20. $\frac{a^3}{b^6}+\frac{u^3}{64v^6}$.
22. $\frac{27}{a^6}+\frac{a^6}{27}$.
24. $1+\frac{64a^3}{27b^3}$.
21. $\frac{n^3}{r^6}+\frac{x^6y^3}{27}$.
23. $\frac{a^3b^6}{c^3d^6}+\frac{64}{27}$.
25. $\frac{n^6r^6}{a^3}+\frac{n^3r^3x^3}{27b^3}$.
26. $\frac{u^3v^6}{w^6}+125$.
29. $\frac{x^6y^9z^9}{a^6b^9}+\frac{u^9y^6z^3}{64a^6}$.
27. $\frac{x^3y^6z^9}{a^9b^9}+\frac{u^3v^6w^9}{a^9b^9}$.
30. $\frac{27a^3x^6}{b^3y^6}+\frac{b^3y^6}{27a^9x^6}$.
28. $\frac{n^3r^6}{1000a^3}+\frac{1000}{a^3}$.
31. $\frac{1000}{a^3b^3}+\frac{1000}{a^3x^3}$.

Sometimes an expression assumes the form of the sum of two cubes, after a factor has been removed from each term, as in the following examples:

32. $a^2x^3+a^5$.
33. $x^3y^3z^2+a^3b^6z^2$.
34. $a^2bxy^3+a^2bx^4z^3$.
35. ax^4yz^3+axy.
36. $\frac{16x^4y^6}{a^3}+\frac{54xy^9z^3}{b^6}$.
37. $5n^2r^9x^3y^6+40n^2x^6z^6$.
38. $2a^2x^6y^6+2a^2$.
39. $uv^2x^6y^3+uv^2w^6$.
40. $128xy^6+54x^4z^9$.
41. $\frac{5a^3}{b^3}+\frac{5x^3}{8y^3z^3}$.

EXERCISE 91.

EXPRESSIONS OF THE FORM $x^4+a^2x^2+a^4$.

183. Some expressions that do not appear to come under any case thus far considered may be made to do so by adding and subtracting some expression. Such is the

case with the expression we are dealing with in this exercise. We notice that the expression here considered, viz.: $x^4+a^2x^2+a^4$, is almost of the form of a perfect square, viz.: the square of x^2+a^2, and, indeed, if the middle term were only $2a^2x^2$, instead of a^2x^2, the expression here considered would be a perfect square. So we *make* the expression a perfect square by adding a^2x^2, but if we add a^2x^2 we must subtract it in order not to change the value of the expression. We may then write,

$$x^4+a^2x^2+a^4=x^4+2a^2x^2+a^4-a^2x^2,$$

or, using a parenthesis, $=(x^4+2a^2x^2+a^4)-a^2x^2$.

Now it is easy to see that we have the difference of two squares, which we have already learned how to factor. Thus we have

$$\begin{aligned} x^4+a^2x^2+a^4&=(x^4+2a^2x^2+a^4)-a^2x^2,\\ &=(x^2+a^2)^2-a^2x^2,\\ &=(x^2+a^2+ax)(x^2+a^2-ax). \end{aligned}$$

EXAMPLES.

Find the factors of each of the following expressions:

1. x^4+9x^2+81.

To make a perfect square of this expression we must add $9x^2$, and if we add $9x^2$ we must of course subtract $9x^2$ afterward. Hence we get

$$(x^4+18x^2+81)-9x^2.$$

We now have the difference of two squares, which, as we have learned before, is equal to the product of the sum multiplied by the difference of the square roots of the two squares. Therefore, we have

$$\begin{aligned} x^4+9x^2+81&=(x^4+18x^2+81)-9x^2,\\ &=(x^2+9+3x)(x^2+9-3x), \end{aligned}$$

or, as is perhaps a more natural arrangement of the terms of the two factors, $(x^2+3x+9)(x^2-3x+9)$.

2. x^4+4x^2+16.

3. $x^4+4x^2y^2+16y^4$.

4. $a^4x^4+4a^2x^2y^2+16y^4$.

5. $1+a^2+a^4$.

6. x^4+x^2+1.

7. $16x^4+4u^2v^4x^2+u^4v^8$.

8. $16x^4+16x^2z^4+16z^8$.
9. $81x^4+36x^2+16$.
10. $16x^4+36x^2y^2+81y^4$.
11. $256+144+81$.
12. $81+9+1$.
13. $81+9a^2+a^4$.
14. $81x^8+36x^4y^4+16y^8$.
15. $x^{12}+4x^6y^4+16y^8$.
16. $16x^{16}+4x^8+1$.
17. $x^4y^8+x^2y^6+y^4$.
18. $x^4y^4+x^2y^6+y^8$.
19. $x^4y^4z^4+4x^2y^2z^2+16$.
20. $\frac{x^4}{a^4}+\frac{9x^2}{a^2}+81$.
21. $\frac{x^4}{a^4}+9x^2+81a^4$.

184. Sometimes an expression assumes the form treated in this exercise, after a factor is removed from each term, as in the following examples:

22. $10mx^4+10a^2mx^2+10a^4m$.
23. $ax^5+a^3x^3+a^5x$.
24. $\frac{a^2x^4}{y^4}+\frac{a^4x^2}{y^2}+a^6$.
25. $10x^4y^4+10x^2y^2+10$.
26. $\frac{2x^8y^4}{z^4}+\frac{2b^4x^4y^2}{z^2}+2b^8$.
27. $48+12a^2b^2+3a^4b^4$.

EXERCISE 92.*

EXPRESSIONS OF THE FORM a^n-b^n.

185. The two general expressions a^n-b^n and a^n+b^n, where n stands for *any* positive whole number, are of so much importance that, although somewhat difficult to factor because of the fact that n is such a general symbol, still we must try to discover under what circumstances expressions have factors of the form $a+b$ or $a-b$. We begin with the first of these two expressions, viz.: a^n-b^n.

* This exercise and the next one are more difficult than most portions of the book, and may, at the discretion of the teacher, be taken now or postponed till some later time in the course.

1. Divide $a^3 - b^3$ by $a - b$ and carefully note the form of the quotient.

2. Divide $a^4 - b^4$ by $a - b$ and carefully note the form of the quotient.

3. Divide $a^5 - b^5$ by $a - b$ and carefully note the form of the quotient.

4. Divide $a^6 - b^6$ by $a - b$ and carefully note the form of the quotient.

5. Without trying, would you think that $a^7 - b^7$ could be divided by $a - b$?

6. Could you guess the form of the quotient of $(a^7 - b^7) \div (a - b)$?

7. Go through the work of division of $(a^7 - b^7) \div (a - b)$ and see if you have guessed the right form of the quotient.

186. Now you would probably guess that $a^n - b^n$ could be divided by $a - b$ and that the quotient would be

$$a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \text{other terms},$$

where the exponent of the power of a continually diminishes and that of the power of b continually increases by 1 in each succeeding term until you arrive at b^{n-1}. A result of this kind where a series of terms are written in regular order but where the number of terms is not definitely known is sometimes written like this:

$$a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + \ldots + b^{n-1}.$$

The dots (read *and so on*) are called the **Sign of Continuation**. They serve to show that the intervening terms are written one after another according to the same law as the terms already written.

8. What would be the term just after $a^{n-4}b^3$?

9. What would be the term just before b^{n-1}?

187. If the student will carefully get fixed in mind the meaning of these dots, we can doubtless go through the work of multiplying and see if this expression multiplied by $a-b$ will produce a^n-b^n; *then*, if this should be the case, what was before a guess becomes a certainty. For the student will remember that the expression we are here talking about, viz.:

$$a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1},$$

was a *guess* at the quotient of dividing a^n-b^n by $a-b$, and therefore if this quotient multiplied by the divisor equals a^n-b^n the guess was right. Now let us multiply

$$\begin{array}{l} a^{n-1}+a^{n-2}b+a^{n-3}b^2+a^{n-4}b^3+\ldots+b^{n-1} \\ a-b \\ \hline a^n+a^{n-1}b+a^{n-2}b^2+a^{n-3}b^3+\ldots\ldots+ab^{n-1} \\ \quad -a^{n-1}b-a^{n-2}b^2-a^{n-3}b^3-a^{n-4}b^4-\ldots-b^n \\ \hline \end{array}$$

Let us stop a moment before adding up these partial products to notice that there is a sign of continuation in each partial product.

10. In the first partial product what is the term just after $a^{n-3}b^3$?

11. What is the term just before ab^{n-1}?

12. In the second partial product what is the term just after $-a^{n-4}b^4$?

13. What is the term just before $-b^n$?

Now adding the above partial products we get the product, which is seen to be a^n-b^n. Therefore,

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}).$$

Therefore, $a-b$ is a factor of a^n-b^n, whatever positive whole number is represented by n.

187. As the sign of continuation usually offers considerable difficulty to the student, let us study a little further the expression $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ +b^{n-1}$ in which this sign first appeared.

14. Is $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ .\ +b^{n-1}$ a factor of a^n-b^n? Why?

15. Is a^n-b^n a multiple of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ .\ +b^{n-1}$?

16. By what expression will we have to multiply $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ .\ +b^{n-1}$ to produce a^n-b^n?

17. Can a^n-b^n be divided by $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ .\ +b^{n-1}$?

18. What is the quotient in example 17?

19. Will any factor of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ +b^{n-1}$ be a factor of a^n-b^n also? Why?

20. In $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ .\ +b^{n-1}$ how many terms contain b?

Write down 5 or 6 terms and then begin at the second term and count the successive terms, looking at the exponent of b while counting, and you can doubtless answer this question.

21. In $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ .\ .\ .\ +b^{n-1}$ how many terms are there altogether?

Notice that each term except the first contains b and you can probably answer this question.

22. The first two terms contain the common factor a^{n-2}. Can their sum be written $a^{n-2}(a+b)$?

23. Do the third and fourth terms have a common factor? What is their H. C. F.? How, then, can their sum be written?

24. Do the fifth and sixth terms have a common factor? What is their H. C. F.? How, then, can their sum be written?

188. If $n=5$, the expression we are considering is $a^4+a^3b+a^2b^2+ab^3+b^4$, and if the terms are grouped together in sets of two as far as possible, we may write $(a^4+a^3b)+(a^2b^2+ab^3)+b^4$, where there is one term left over at the end which cannot be put in any group because there is no term to go with it. But if $n=6$, the expression we are considering is $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$, and if the terms of this are grouped in sets of two as far as possible, we may write $(a^5+a^4b)+(a^3b^2+a^2b^3)+(ab^4+b^5)$, where there is *no* term left over at the end.

25. If in $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ the terms be grouped together in sets of two as far as possible, when will there be one term left over at the end and when will all the terms be thus grouped?

Answer. All the terms can be thus grouped when there is an *even* number of terms, *i. e.* when n stands for an even number, and one term will be left over when the number of terms is odd, *i. e.* when n stands for an odd number.

26. Suppose n an even number, group all the terms of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ in sets of two, take out the H. C. F. from each group, and then tell what is a factor of the whole expression.

Notice that the factor which you have just found is a factor of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ *only* when n is an *even* number.

27. Is $a+b$ a factor of a^n-b^n when n is an even number?

See question 19.

189. Now the student can probably understand the following demonstration:

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}.$$

Hence, $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1})$.

The number of terms of this expression, $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ is n, for there are $n-1$ terms that

contain b (as can be seen by beginning at the second term and counting, noticing the exponents of b while counting,) and one term that does not contain b, so there are n terms in all. Therefore, when n stands for an even number the terms of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ can be grouped in sets of two, and in each set we can take out the H. C. F. and express the result as follows:

$$a^{n-2}(a+b)+a^{n-4}b^2(a+b)+a^{n-6}b^4(a+b)+\ldots+b^{n-2}(a+b)$$

Evidently $a+b$ is a factor of this expression, *i. e.* $a+b$ is a factor of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$.

But any factor of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ is also a factor of a^n-b^n because a^n-b^n is a multiple of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$.

Therefore $a+b$ is a factor of a^n+b^n when n stands for any *even* number.

Notice, this demonstration will not hold if n is an odd number because the terms of $a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots+b^{n-1}$ cannot then *all* be grouped in sets of two, for one term will remain over.

190. We have so far reached the following results: a^n-b^n can *always* be divided by $a-b$, and a^n-b^n can be divided by $a+b$ *when n is any even number*, or stated in another way, $a-b$ is *always* a factor of a^n-b^n and $a+b$ is a factor of a^n-b^n *when n is any even number.*

191. Thus it appears that the difference between like powers of two numbers can always be factored, but it must not be supposed from what has been said that the easiest and best way to factor a^n-b^n is always to take out the factor $a-b$ first, for it may be that the remaining expression after this factor has been removed will be harder to factor than the one we started with would be, if we proceed to take out some other factor first. This will be fully seen in the examples which follow.

EXAMPLES.

Factor as far as you can each of the following expressions :

1. a^4-b^4.

This we know has a factor $a-b$, and also because the exponent is even a factor $a+b$, and if we take out each of these factors in turn we would have left a^2+b^2.

Therefore, $a^4-b^4=(a-b)(a+b)(a^2+b^2)$.

Or we might proceed thus: a^4-b^4 may be regarded as the difference of two squares and factor accordingly.

Therefore, $a^4-b^4=(a^2)^2-(b^2)^2=(a^2-b^2)(a^2+b^2)$
$=(a-b)(a+b)(a^2+b^2)$,

the same as before, as it ought to be.

2. x^4-1.

3. x^4-y^4.

4. $x^4y^4-z^4$.

5. $r^4s^4-t^4$.

6. $16x^4-81$.

7. $1-x^4$.

8. $1-x^4y^4$.

9. $16a^4-x^4y^4$.

10. $16x^4-16y^4$.

11. $x^4y^4-u^4v^4$.

12. $\frac{x^4}{y^4}-z^4$.

13. $\frac{u^4}{v^4}-\frac{x^4}{16}$.

14. a^5-b^5.

From the general discussion which precedes we know that the factors of this are $a-b$ and $a^4+a^3b+a^2b^2+ab^3+b^4$, and this is the only way we have at present to factor this expression.

15. x^5-1.

16. x^5-y^5.

17. $x^5y^5-z^5$.

18. u^5v^5-32.

19. $1-x^5$.

20. $243-32$.

21. $32x^5y^5-u^5v^5w^5$.

22. $\frac{x^5y^5}{z^5}-\frac{r^5t^5}{w^5}$.

23. $a^5x^5y^5-32u^5v^5w^5$.

24. $\frac{a^5x^5}{y^5}-\frac{u^5v^5}{y^5}$.

25. a^6-b^6.

This may be regarded as the difference of the two cubes, and hence we may write

$$a^6-b^6=(a^2)^3-(b^2)^3=(a^2-b^2)(a^4+a^2b^2+b^4).$$

Each of these factors is a form already treated, and so we know the factors of each factor, viz:

$$a^2 - b^2 = (a-b)(a+b)$$

and $$a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab).$$

Therefore, $$a^6 - b^6 = (a-b)(a+b)(a^2 + b^2 + ab)(a^2 + b^2 - ab).$$

Or, if we prefer, we may regard $a^6 - b^6$ as the difference of two squares, and hence may write:

$$a^6 - b^6 = (a^3)^2 - (b^3)^2 = (a^3 - b^3)(a^3 + b^3).$$

Here again each factor is a form already treated, and so we know the factors of each factor, viz:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

and $$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Therefore, $$a^6 - b^6 = (a-b)(a+b)(a^2 + ab + b^2)(a^2 - ab + b^2).$$

This agrees with the result obtained before, as it ought to do.

26. $x^6 - 1$.

28. $\dfrac{x^6}{y^6} - u^6 x^6$.

30. $x^6 y^6 z^6 - 1$.

27. $x^6 y^6 - z^6$.

29. $n^6 x^6 - r^6 y^6 z^6$

31. $\dfrac{x^6}{y^6} - \dfrac{y^6}{x^6}$.

32. $a^7 - b^7$.

From the general discussion which preceeds we know that the factors of this are $a-b$ and $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$, and these are the only factors of $a^7 - b^7$ that we can find now

33. $x^7 y^7 - 1$.

35. $x^7 y^7 - z^7$.

34. $\dfrac{x^7}{y^7} - \dfrac{u^7}{v^7}$.

36. $1 - \dfrac{u^7 v^7}{w^7 x^7}$.

37. $a^8 - b^8$.

This may be considered either as the difference of two fourth powers, or the difference of two squares. Taking it in the latter way we may write,

$$a^8 - b^8 = (a^4 - b^4)(a^4 + b^4).$$

The first of these factors has been considered before, so we know how to factor it. Therefore we have,

$$\begin{aligned} a^8 - b^8 &= (a^4 - b^4)(a^4 + b^4) \\ &= (a^2 - b^2)(a^2 + b^2)(a^4 + b^4) \\ &= (a-b)(a+b)(a^2 + b^2)(a^4 + b^4). \end{aligned}$$

38. x^8y^8-1.

39. $\frac{x^8}{y^8}-\frac{u^8}{v^8}$.

40. $x^8-u^8v^8$.

41. $\frac{x^8y^8}{z^8}-\frac{u^8}{v^8}$.

42. a^9-b^9.

This may be regarded as the difference of two cubes, viz:

$$(a^3)^3-(b^3)^3.$$

Therefore,
$$a^9-b^9=(a^3-b^3)(a^6+a^3b^3+b^6)$$
$$=(a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6).$$

43. $x^9y^9-z^9$.

44. $\frac{x^9}{y^9}-1$.

45. $u^9v^9-s^9t^9$.

46. $\frac{u^9}{v^9}-\frac{v^9}{u^9}$.

EXERCISE 93.

EXPRESSIONS OF THE FORM a^n+b^n.

192. In the previous exercise we found that a^n-b^n was always divisible by $a-b$. Now this dividend, a^n-b^n, means that the nth power of b is to be subtracted from the nth power of a, and the divisor, $a-b$, means that the number represented by b is to be subtracted from the number represented by a, and in both dividend and divisor a and b may stand for either positive or negative numbers. We may then write *any* numbers or letters we like in place of either a or b or both.

Suppose, then, we take the case where n stands for an *odd* number and write $-b$ in place of b in both dividend and divisor. Now, because n is an odd number, $(-b)^n=-b^n$, and if this expression, $-b^n$, be subtracted from a^n we get a^n+b^n for a dividend, and if $-b$ be subtracted from a we get $a+b$ for a divisor.

Therefore, $a+b$ is a divisor, *i. e.*, a *factor*, of a^n+b^n *when n is any odd number.*

193. To find out whether $a+b$ or $a-b$ is a factor of a^n+b^n when n is *even*, a little preliminary study is necessary.

1. Is 5 a factor of 55? Is 5 a factor of 15? Is 5 a factor of $55+15$? Is 5 a factor of $55-15$?

2. Is a a factor of ax? Is a a factor of ay? Is a a factor of $ax+ay$? Is a a factor of $ax-ay$?

3. If a number is a factor of each of two other numbers. is it a factor of the sum of those numbers? Is it a factor of the difference of those numbers?

4. If one number is a factor of a second and not of a third, can the first number be a factor of the sum of the second and third numbers? Can the first number be a factor of the difference of the second and third numbers?

5. When n is even is $a+b$ a factor of a^n-b^n? When n is even is $a+b$ a factor of $2b^n$? When n is even is $a+b$ a factor of $(a^n-b^n)+2b^n$?

6. Since $(a^n-b^n)+2b^n=a^n+b^n$ is $a+b$ a factor of a^n+b^n when n is even?

7. Is $a-b$ a factor of a^n-b^n? Is $a-b$ a factor of $2b^n$? Is $a-b$ a factor of $(a^n-b^n)+2b^n$?

8. Since $(a^n-b^n)+2b^n=a^n+b^n$ is $a-b$ a factor of a^n+b^n?

194. Questions 6 and 8 if rightly answered and understood, give us the fact, that a^n+b^n is never divisible by $a-b$ and a^n+b^n is not divisible by $a+b$ when n is even.

195. We previously found that a^n+b^n *is* divisible by $a+b$ when n is an *odd* number.

196. What has been found in the previous exercise and in this one, may now be stated as follows:

a^n-b^n is always divisible by $a-b$.

a^n-b^n is divisible by $a+b$ when n is an even number, but not when n is an odd number.

a^n+b^n is divisible by $a+b$ when n is an odd number, but not when n is an even number.

a^n+b^n is never divisible by $a-b$.

197. Although a^n+b^n cannot be divided by either $a+b$ or $a-b$ when n is even, yet it is not stated that in this case a^n+b^n has *no* factors, for sometimes other factors can be found, as will be seen by the examples to follow and the explanations accompanying them.

Factor each of the following expressions:

9. a^5+b^5.

By the general discussion we know that this has a factor $a+b$, and another factor, found by division of a^5+b^5 by $a+b$, is

$$a^4-a^3b+a^2b^2-ab^3+b^4.$$

10. x^5+1. 12. $a^5x^5+b^5y^5$. 14. $n^5x^5+32x^5y^5z^5$.

11. $x^5y^5+z^5$. 13. $1+32x^5$. 15. $a^5+32u^5v^5w^5$.

16. $\dfrac{x^5}{y^5}+\dfrac{32}{a^5}$. 18. $\dfrac{a^5x^5}{y^5}+\dfrac{b^5z^5}{y^5}$.

17. $\dfrac{a^5}{b^5}+\dfrac{b^5}{a^5}$. 19. $1+\dfrac{a^5}{32}$.

20. a^6+b^6.

This may be regarded as the sum of two cubes, and therefore we may write $a^6+b^6=(a^2)^3+(b^2)^3=(a^2+b^2)(a^4-a^2b^2+b^4)$.

21. x^6+y^6. 23. $a^6x^6+64b^6y^6$. 25. $64+x^6y^6z^6$.

22. x^6+1. 24. u^6v^6+64. 26. $x^6y^6z^6+x^6y^6w^6$.

27. $\dfrac{x^6}{y^6}+1$. 28. $\dfrac{x^6}{y^6}+\dfrac{y^6}{x^6}$.

29. a^7+b^7.

By the general discussion we know that this has a factor $a+b$, and the other factor obtained by division will be found to be

$$a^6-a^5b+a^4b^2-a^3b^3+a^2b^4-ab^5+b^6.$$

30. x^7+y^7. 32. $u^7v^7+x^7y^7$. 34. $1+x^7y^7z^7$.

31. u^7+1. 33. $a^7b^7c^7+128$. 35. $128+1$.

36. $\frac{u^7}{v^7}+\frac{x^7}{y^7}$. 37. $\frac{x^7}{y^7}+\frac{y^7}{x^7}$.

38. a^9+b^9.

This may be regarded as the sum of two cubes, and hence we may write $a^9+b^9=(a^3)^3+(b^3)^3=(a^3+b^3)(a^6-a^3b^3+b^6)$,

$$=(a+b)(a^2-ab+b^2)(a^6-a^3b^3+b^6).$$

39. x^9+1. 41. $u^9+v^9w^9x^9$. 43. $\frac{x^9}{y^9}+\frac{u^9}{v^9}$.

40. $x^9y^9+z^9$. 42. $a^9x^9+b^9y^9z^9$. 44. $\frac{x^9}{y^9}+\frac{y^9}{x^9}$.

45. $a^{10}+b^{10}$.

This may be regarded as the sum of two fifth powers, and hence we may write,

$$a^{10}+b^{10}=(a^2)^5+(b^2)^5=(a^2+b^2)(a^8-a^6b^2+a^4b^4-a^2b^6+b^8).$$

46. $x^{10}+y^{10}$. 48. $1024+x^{10}y^{10}$. 50. $\frac{a^{10}}{b^{10}}+\frac{x^{10}}{y^{10}}$.

47. $a^{10}+1$. 49. $1024+1$. 51. $\frac{x^{10}}{y^{10}}+1$.

198. In the expressions considered in this exercise and the preceding, the exponent in each term is the same, but the methods used enable us in some cases to find factors of an expression in which the exponents are not the same in each term, as for instance in the expression a^5+b^{10}. The exponents are 5 and 10 respectively, but we may consider $b^{10}=(b^2)^5$ and, therefore, a^5+b^{10} may

be considered the sum of two fifth powers, viz: $a^5+(b^2)^5$, and may be factored as in example 9, using b^2 the same as b was used before.

We add a few miscellaneous examples on the last two exercises, where in some cases a factor must be removed from each term before the expression is in the form treated in either of these last two exercises.

52. $3a^4-48$.

53. $2a^6-32a^2x^4$.

54. $3x^4-3y^{12}$.

55. $x^5+a^5y^{10}$.

56. $3u^5v^5-3u^5x^{10}$.

57. $\frac{4}{(x^3+y^3)^6}-\frac{4}{(x^3-y^3)^6}$.

58. $(a+b)^8-(a^2-b^2)^8$.

59. $(a^2+b^2)^4-(a^2-b^2)^4$.

60. $\frac{(x+y)^5}{32}-\frac{(x+y)^{10}}{32}$.

61. $x^6-a^6y^{12}$.

62. $5x^{12}-5x^6y^{12}$.

63. $10a^8-10b^8c^8$.

64. $(x+y)^5-(x-y)^5$.

65. $3(x^3+y^3)^3-3(x^3-y^3)^3$

66. $\frac{(x^3-y^3)^6}{64}-\frac{64}{(x^3-y^3)^6}$.

67. $a^{18}-a^{12}$.

68. $3mx^{14}-3m^8y^7$.

69. $2a^6-\frac{2}{a^6}$.

EXERCISE 94.

MISCELLANEOUS FACTORS.

199. Some expressions are quite difficult to factor unless one sees how the expression may be changed in form by rearranging or grouping of the terms, or by adding and then subtracting the same expression, or by some other device to change the expression into a form that will be recognized as coming under some case already treated. To help the student see some of the devices, we take some of these irregular expressions and work

them out. The student is advised to go through the explanation given, to see just *what* is done, and if possible, *why* it is done, and after two or three cases to cover up the explanation given and see if some method suggests itself; if not, see how the work is *started* and then try to complete it without looking at the rest of the explanation. If still unable to do the work, study the whole explanation given.

1. Let us factor $a^3+b^3+c^3+3b^2c+3bc^2$.

We may arrange the terms thus:

$$a^3+b^3+3b^2c+3bc^2+c^3$$

where the last four terms form a perfect cube, viz: the cube of $b+c$. Therefore, the given expression may be written $a^3+(b+c)^3$.

We now have the sum of two cubes, and therefore the factors are $a+b+c$ and $a^2-a(b+c)+(b+c)^2$, or $a+b+c$, and $a^2+b^2+c^2-ab-ac+2bc$.

Hence, $a^3+b^3+c^3+3b^2c+3bc^2$

$$=(a+b+c)(a^2+b^2+c^2-ab-ac+2bc).$$

2. Let us factor $a^3+b^3+c^3+3a^2b+3ab^2$.

We may arrange the terms thus:

$$a^3+3a^2b+3ab^2+b^3+c^3,$$

where the first four terms form a perfect cube, viz: the cube of $a+b$. Therefore, the given expression may be written $(a+b)^3+c^3$.

We now have the sum of two cubes, and therefore, the factors are $a+b+c$ and $(a+b)^2-(a+b)c+c^2$, or $a+b+c$, and $a^2+b^2+c^2+2ab-ac-bc$.

Hence, $a^3+b^3+c^3+3a^2b+3ab^2$

$$=(a+b+c)(a^2+b^2+c^2+2ab-ac-bc).$$

3. Let us factor the expression $a^4+2a^3b-a^2-2ab^3+b^2-b^4$.

This can be arranged thus:

$$(a^4-b^4)-(a^2-b^2)+2ab(a^2-b^2),$$

where it is plain that a^2-b^2 is a factor of each of the three parts into which the expression is grouped. Taking out this factor, the expression may be written

$$[a^2-b^2][(a^2+b^2)-1+2ab],$$

or $$(a^2-b^2)[(a^2+2ab+b^2)-1],$$

or $$(a^2-b^2)[(a+b)^2-1].$$

Each of these factors is itself the difference of two squares, and hence each may be further factored thus:

$$a^2-b^2=(a-b)(a+b),$$

$$(a+b)^2-1=(a+b-1)(a+b+1).$$

Therefore we may write,

$$a^4+2a^3b-a^2-2ab^3+b^2-b^4$$
$$=(a-b)(a+b)(a+b-1)(a+b+1).$$

4. Let us factor the expression $a^2x^2-b^2x^2+b^2-a^2$.

This may be written thus:

$$(a^2-b^2)x^2+b^2-a^2,$$

or $$(a^2-b^2)x^2-(a^2-b^2),$$

or $$(a^2-b^2)(x^2-1),$$

or $$(a-b)(a+b)(x-1)(x+1).$$

5. Let us factor the expression $a^4-a^2+b^4-b^2-2a^2b^2-2ab$.

This may be written thus:

$$(a^4-2a^2b^2+b^4)-(a^2+2ab+b^2),$$

or $$(a^2-b^2)^2-(a+b)^2.$$

This is the difference of two squares, and hence may be written as the product of the sum into the difference of the two numbers which are squared. Hence,

$$a^4-a^2+b^4-b^2-2a^2b^2-2ab$$
$$=[(a^2-b^2)-(a+b)][(a^2-b^2)+(a+b)]$$
$$=[(a+b)(a-b)-(a+b)][(a+b)(a-b)+(a+b)]$$
$$=(a+b)(a-b-1)(a+b)(a-b+1).$$

6. Let us factor $a^3+b^3+c^3-3abc$.

This may be written thus:

$$a^3+(b^3+3b^2c+3bc^2+c^3)-3b^2c-3bc^2-3abc,$$

or
$$a^3+(b+c)^3-3bc(a+b+c).$$

The first two terms are now the sum of two cubes, and hence contain the factor $a+b+c$, and this is also a factor of the last term, as is very evident. We may therefore write the expression considered in the form

$$(a+b+c)[a^2-a(b+c)+(b+c)^2]-3bc(a+b+c),$$

or
$$(a+b+c)(a^2-ab-ac+b^2-bc+c^2).$$

7. Let us factor $a^3+b^3+c^3+ab^2+ac^2+a^2b+a^2c+bc^2+b^2c$.

This may be written thus:

$$(a^3+a^2b+a^2c)+(b^3+ab^2+b^2c)+(c^3+ac^2+bc^2).$$

In the first group a^2 is a factor of each term, in the second group b^2 is a factor of each term, and in the third group c^2 is a factor of each term. Hence, we may write

$$a^2(a+b+c)+b^2(a+b+c)+c^2(a+b+c).$$

And now evidently $a+b+c$ is a common factor throughout. Hence, the original expression equals

$$(a^2+b^2+c^2)(a+b+c).$$

8. $12x^2-36x+24$.

9. $5x^2+5x-10$.

10. $x^2y+xy-x^2-x-2y+2$.

11. $(a+b+c)^2-(a-b-c)^2$.

12. $(c+d)^3+(c-d)^3$.

13. x^2-4y^2+x-2y.

14. x^3+8y^3+x+2y.

15. $a^5-8a^2b^3$.

16. $500x^2y-20y^3$.

17. $1-a^2x^2-b^2y^2+2abxy$.

18. $a^2+x^2-(y^2+z^2)-2(yz-ax)$.

19. $5x^4-15x^3-90x^2$.

EXERCISE 95.

H. C. F. of Expressions which Can be Factored.

200. The highest common factor of several expressions has already been defined, (see Art. 82,) and a method of finding the H. C. F. of several *monomial* expressions has been given. We now take up the H. C. F. of polynomials.

201. The first method to consider is exactly the same as that presented in the case of monomials, viz.:

Resolve each expression into its prime factors and take the product of all those which are common to all the expressions.

By this method we can readily find the H. C. F. of any expressions which can be readily resolved into their prime factors.

Examples.

Find the H. C. F. of each of the following sets of expressions:

1. x^2+xy and x^2-y^2.

2. $2x^2-2xy$ and $3x^2-3y^2$.

3. x^3-y^3 and x^2-y^2.

4. x^3+y^3 and x^2-y^2.

5. x^3+x^2y and x^3+y^3.

6. $12a^3x^2y-4a^3xy^2$ and $30a^2x^3y^2-10a^2x^2y^3$.

7. $8a^3b^2c-12a^2bc^3$ and $6ab^4c+4ab^3c^2$.

8. x^2-2x-3 and x^2+x-12.

9. $2a(a^2-b^2)$ and $4b(a-b)^2$.

10. $3x^3+6x^2-24x$ and $6x^3-96x$.

11. $x^2-9x-10$ and x^2+4x+3.

12. x^2-4 and x^2-5x+6.

13. a^6-b^6 and a^4-b^4.

14. x^2+x-6 and x^2-3x+2

15. x^2+x+1 and x^3-1.

16. $2x^3+2$, $3x^2-3$, and x^2+3x+2.

17. x^2-3x+2, x^2-6x+8, and x^2+x-6.

18. $x^2y^2-z^2$, $x^2y^2-3xyz+2z^2$, and $x^3y^4-yz^3$.

19. x^4-8x^2+16 and $x^3y^3+4x^2y^3+4xy^3$.

20. x^3-x^2+x-1, $x^4+x^3-x^2-x$, and x^2+2x-3.

EXERCISE 96.

H. C. F. OF EXPRESSIONS NOT EASILY FACTORED.

202. The method used in the preceding exercise for finding the H. C. F. is not appropriate when the expressions given are not easily factored. In case the expressions given are not easily factored, the method to be pursued depends upon a few principles which must be understood before the method is given.

1. If an expression be multiplied by some number or expression, does the product contain all the factors the original expression contained?

2. Does the product contain any factors the original expression did not contain?

3. Are the factors of an expression also factors of any multiple of that expression?

4. If one expression is a factor of two other expressions, will it be a factor of the sum of those expressions?

5. Will it be a factor of the difference of those expressions?

6. Will it be a factor of any multiple of the first expression plus any multiple of the second expression?

7. Will it be a factor of any multiple of the first expression minus any multiple of the second expression?

203. If these questions are understood, we may state the two principles to be used in finding the H. C. F. as follows:

I. Any factor of an expression is a factor of any multiple of that expression.

II. Any common factor of two expressions is a factor of the sum or difference of those expressions or of the sum or difference of any multiples of those expressions.

204. Our problem is to find, not merely a *common* factor, but the *highest* common factor of two or more expressions. This problem can often be considerably simplified by *first* taking out from each of the given expressions all *monomial* factors and finding the H. C. F. of these factors, if there be any, and then find the H. C. F. of the remaining portions of the given expressions. After this has been done we must multiply the H. C. F. of the monomial factors by the H. C. F. of the remaining polynomial factors, when we will have the whole of the H. C. F. of the given expressions.

Let us find the H. C. F. of

$$x^3-x^2-4 \qquad (1)$$

and

$$3x^3+x^2-4x-20. \qquad (2)$$

Neither of these expressions have any *monomial* factors, therefore the H. C. F. of them will have no monomial factors.

8. Will any factor of expression (1) be a factor of 3 times this expression, *i. e.*, of

$$3x^3-3x^2-12? \qquad (3)$$

9. Will any common factor of (1) and (2) be a common factor of (2) and (3)?

10. Will the H. C. F. of (1) and (2) be the H. C. F. of (2) and (3)?

11. Will any common factor of (2) and (3) be a factor of their difference, which is

$$4x^2-4x-8? \qquad (4)$$

12. Will the H. C. F. of (2) and (3) be the H. C. F. of (3) and (4)?

13. Will the H. C. F. of (1) and (2) be the H. C. F. of (3) and (4)?

Now remember that the H. C F. of (1) and (2) can have no *monomial* factors, and therefore the H. C. F. of (3) and (4) can have no monomial factors, and the next question can be answered.

14. Will the H. C. F. of (3) and (4) be the H. C. F. of

$$x^3-x^2-4 \qquad (5)$$

and

$$x^2-x-2? \qquad (6)$$

15. Will the H. C. F. of (1) and (2) be the H. C. F. of (5) and (6)?

16. Will any factor of (6) be a factor of x times this expression, which is

$$x^3-x^2-2x? \qquad (7)$$

17. Will the H. C. F. of (5) and (6) be the H. C. F. of (5) and (7)?

18. Will any common factor of (5) and (7) be a factor of their difference, which is

$$2x-4\,?\qquad(8)$$

19. Will the H. C. F. of (5) and (7) be a factor of (8)?

20. Will the H. C. F. of (5) and (7) be the H. C. F. of (7) and (8)?

21. Will the H. C. F. of (7) and (8) be the H. C. F. of

$$x^2-x-2\qquad(9)$$

and

$$x-2\,?\qquad(10)$$

22. Will the H. C. F. of (9) and (10) be the H. C. F. of (5) and (7)?

23. Will the H. C. F. of (9) and (10) be the H. C. F. of (5) and (6)?

24. Will the H. C. F. of (9) and (10) be the H. C. F. of (1) and (2)?

25. What is the H. C. F. of (9) and (10)?

26. What is the H. C. F. of (1) and (2)?

205. Let us now look over and see how these various expressions came about. Expression (1) was multiplied by 3 to get expression (3). Where did this multiplier 3 come from? We notice that if we divide (2) by (1) the quotient is 3 and the remainder is $4x^2-4x-8$, which is numbered (4). This expression (4) contains a *monomial* factor 4, which when rejected leaves x^2-x-2, the expression numbered (6). The work so far can be arranged as follows:

$$\begin{array}{rl} x^3-x^2-4 \;)\; 3x^3+x^2-4x-20 \;(\; 3 \\ \underline{3x^3-3x^2\qquad -12} \\ 4 \;)\; 4x^2-4x-8 \\ \underline{\qquad\qquad} \\ x^2-x-2 \end{array}$$

We now deal with (5) and (1), or what is the same thing, (5) and (6). These two expressions have the same H. C. F. as (1) and (2) and are easier to deal with because the highest exponents in these two expressions are 2 and 3 instead of 3 and 3 as before. Thus our problem is a little simpler than before.

Expression (6) was multiplied by x to produce expression (7). But where did this multiplier x come from? Notice that if we divide expression (5) by expression (6) the quotient is x and the remainder is $2x-4$, which is the expression numbered (8). This expression (8) contains the *monomial* factor 2, which when rejected leaves $x-2$, the expression numbered (10). This second part of the work can be arranged as follows:

$$\begin{array}{l} x^2-x-2 \;)\; x^3-x^2 \qquad -4 \;(\; x \\ \qquad\qquad \underline{x^3-x^2-2x} \\ \qquad\qquad\qquad 2 \;)\; \underline{2x-4} \\ \qquad\qquad\qquad\qquad\quad x-2 \end{array}$$

We now deal with (9) and (10) instead of (5) and (6). These two expressions, (9) and (10), have the same H. C. F. as (5) and (6) and are easier to deal with because in these the highest exponents are 1 and 2 respectively instead of 2 and 3 as before. We can even find the H. C. F. of these expressions, (9) and (10), by the method previously used, or we can keep on with the process so far employed and divide expression (10) by expression (9). This third part of the work can be arranged as follows:

$$\begin{array}{l} x-2 \;)\; x^2- \;\; x-2 \;)\; x+1 \\ \qquad\quad \underline{x^2-2x} \\ \qquad\qquad\quad x-2 \\ \qquad\qquad\quad \underline{x-2} \end{array}$$

As this division is exact, the H. C. F. of (9) and (10) is $x-2$.

206. Thus it appears that the process of finding the H. C. F. of two expressions, when they cannot be readily factored, is *to divide one expression by the other, the divisor by the remainder, the last divisor by the last remainder, and so on until there is no remainder. The last divisor is the H. C. F.*

It must be remembered that this process is not to be employed until all the monomial factors are first taken out of the given expression, so that the H. C. F. of the *remaining portions* of the given expressions contains no monomial factors. This being done, we may, at any stage of the process just described, remove from any dividend or divisor any monomial factor we please; and it will usually be best to remove them whenever we can.

All the work of the preceding example may be arranged as follows:

$$
\begin{array}{r}
x^3-x^2-4 \;)\; 3x^3+\;x^2-4x-20 \;(\; 3 \\
\underline{3x^3-3x^2\qquad -12} \\
4 \;)\; 4x^2-4x-\;8 \\
\underline{\qquad\qquad\quad} \\
x^2-\;x-\;2
\end{array}
$$

$$
\begin{array}{r}
x^2-x-2 \;)\; x^3-x^2\qquad -4 \;(\; x \\
\underline{x^3-x^2-2x\qquad\quad} \\
2 \;)\; 2x-4 \\
\underline{\qquad\quad} \\
x-2
\end{array}
$$

$$
\begin{array}{r}
x-2 \;)\; x^2-\;x-2 \;(\; x+1 \\
\underline{x^2-2x\qquad} \\
x-2 \\
\underline{x-2\qquad} \\
\qquad
\end{array}
$$

Let us find the H. C. F. of

$$4x^3-3x^2-24x-9$$

and

$$8x^3-2x^2-53x-39.$$

As there are no monomial factors, we may arrange the work thus:

$$4x^3-3x^2-24x-9 \;)\; 8x^3-2x^2-53x-39 \;(\; 2$$
$$\underline{8x^3-6x^2-48x-18}$$
$$4x^2-\;5x-21$$

$$4x^2-5x-21 \;)\; 4x^3-3x^2-24x-9 \;(\; x$$
$$\underline{4x^3-5x^2-21x}$$
$$2x^2-\;3x-9$$

$$2x^2-3x-9 \;)\; 4x^2-5x-21 \;(\; 2$$
$$\underline{4x^2-6x-18}$$
$$x-\;3$$

$$x-3 \;)\; 2x^2-3x-9 \;(\; 2x+3$$
$$\underline{2x^2-6x}$$
$$3x-9$$
$$\underline{3x-9}$$

207. It will sometimes happen that the numerical coefficients are such that the first term of the expression used for a dividend is not exactly divisible by the first term of the divisor, in which case we multiply the dividend by the smallest number that will make the division exact. This peculiarity is illustrated by the following example:

Find the H. C. F. of

$$3a^3-4a^2+3a-2$$

and

$$2a^4-3a^3+a^2+a-1.$$

Here we use the first expression for a divisor and the second for a dividend, and evidently the first term, $2a^4$, of the dividend is not exactly divisible by $3a^3$, the first term of the divisor, so we multiply the dividend by 3; and the work may be arranged thus:

$$
\begin{array}{r}
2a^4-3a^3+\ a^2+\ a-1 \\
3 \\
\hline
3a^3-4a^2+3a-2\ )\ 6a^4-9a^3+3a^2+3a-3\ (\ 2a \\
6a^4-8a^3+6a^2-4a \\
\hline
-\ a^3-3a^2+7a-3
\end{array}
$$

In a case like this, where the first term of the remainder has a minus sign, we change all the signs before using it as a divisor. This is equivalent to multiplying by -1. Making the change, we continue as follows:

$$
\begin{array}{r}
a^3+3a^2-7a+3\ )\ 3a^3-\ 4a^2+\ 3a-\ 2\ (\ 3 \\
3a^3+\ 9a^2-21a+\ 9 \\
\hline
-13a^2+24a-11
\end{array}
$$

Here again the first term of the remainder having a minus sign, we change all the signs of the remainder before using it as a divisor; but even then, the first term of the expression we are to use as a divisor not being divisible by $13a^2$, we multiply the dividend (which was the divisor in the operation just performed) by 13, so that the division will be exact, and continue as follows:

$$
\begin{array}{r}
a^3+\ 3a^2-\ \ 7a+\ 3 \\
13 \\
\hline
13a^2-24a+11\ )\ 13a^3+39a^2-\ 91a+39\ (\ a+4 \\
13a^3-24a^2+\ 11a \\
\hline
63a^2-102a+39 \\
52a^2-\ 96a+44 \\
\hline
11a^2-\ \ 6a-\ 5
\end{array}
$$

Again, when we use this remainder for divisor and this divisor for dividend, the first term of the new dividend is not exactly divisible by the first term of the new divisor, so we multiply again by such a number that the division will be exact and proceed as follows:

$$
\begin{array}{r}
13a^2-\ 24a+\ 11 \\
11 \\
\hline
11a^2-6a-5\ )\ 143a^2-264a+121\ (\ 13 \\
143a^2-\ 78a-\ 65 \\
\hline
-186a+186
\end{array}
$$

Before using this remainder as a divisor we will take out the factor -186, leaving $a-1$, and then proceed as follows:

$$
\begin{array}{r}
a-1\ )\ 11a^2-\ 6a-5\ (\ 11a+5 \\
11a^2-11a \\
\hline
5a-5 \\
5a-5 \\
\hline
\end{array}
$$

Since this division is exact, we conclude that $a-1$ is the H. C. F. of the expressions we started with. This is an unusually hard example, and the student who can follow this will not find any trouble with any example given.

Examples.

Find the H. C. F. of the following expressions:

27. $5x^2+4x-1$ and $20x^2+21x-5$.
28. $2x^3-4x^2-13x-7$ and $6x^3-11x^2-37x-20$.
29. $x^3+4x^2-5x-20$ and $x^3+6x^2-5x-30$.
30. $2x^3-8x^2+8x$ and $3x^4-6x^3-9x^2+18x$.
31. $3a^3-22a-15$ and $5a^4+a^3-54a^2+18a$.
32. x^3-2x^2-x+2 and $x^3-6x^2+11x-6$.
33. $3xy(x^3-x^2+x+3)$ and $4y^2(x^4+x^3-3x^2-x+2)$.
34. x^3-2x^2+2x-1 and $x^4-3x^3+2x^2+x-1$.
35. $x^3-8xy^2-3y^3$ and $x^3-4x^2y+9y^3$.
36. x^3-8x+3 and x^6+3x^5+x+3.

37. $x^3+10x^2+33x+36$ and $x^3+9x^2+23x+15$.

38. x^3-x^2-4x+4 and $2x^3-x^2-6x+8$.

39. $2x^3-5x^2-x+6$ and $4x^3-x^2-11x-6$.

40. $a^2+5ab+4b^2$ and $a^3+4a^2b+5ab^2+2b^3$.

41. $4m^3-3m+1$ and $8m^4+m-1$.

42. $2a^3+a^2-2a-6$ and $6a^3-a^2-14a+3$.

43. $8a^3-8a^2-4a-3$ and $2a^4+3a^3-3a^2-7a-3$.

44. x^5-x^2-x-1 and $2x^5+x^2-2x+1$.

45. $2x^3+x^2+2x-12$ and $2x^3-7x^2+14x-12$.

46. a^4+67a^2+66 and $a^4+2a^3+2a^2+2a+1$.

47. $28a^2+37a-21$ and $35a^2+62a-33$.

48. $7m^3-13m^2+34m-72$ and $7m^3-6m^2+35m-36$.

49. $2x^3-3ax^2-7a^2x+4a^3$ and $6x^3-7ax^2-4a^2x+3a^3$

50. $2x^3+5x^2y+2xy^2-y^3$ and $3x^3+2x^2y+xy^2+2y^3$.

51 $2x^3-6x^2-2x+6$ and $3x^4-9x^2+6$.

52. $x^4-x^3-7x^2+x+6$ and $x^4+x^3-7x^2-x+6$.

53. $3x^3-4x^2-x-14$ and $6x^3-11x^2-10x+7$.

54. x^5-x^3-x-1 and $x^7-x^6-x^4+1$.

55. $10x^3+25ax^2-5a^3$ and $4x^3+9ax^2-2a^2x-a^3$.

56. $3x^4-8x^3y+5x^2y^2-2xy^3$ and $9x^4+2x^2y^2+y^4$.

208. When we wish to find the H. C. F. of more than two expressions, we first find the H. C. F. of any two of them and then find the H. C. F. of this result and the third expression, and so on until all the expressions are used. The final result is the H. C. F. of all the given expressions.

EXAMPLES.

Find the H. C. F. of the following expressions:

57. $2x^2+3x-5$, $3x^2-x-2$, and $2x^2+x-3$.

58. a^3-3a-2, $2a^3+3a^2-1$, and a^3+1.

59. $12(a^4-b^4)$, $10(a^6-b^6)$, and $8(a^4b-ab^4)$.

60. x^2-x-12, x^2+6x+8, and x^3-4x^2-x+4.

61. $x^3-6x^2+11x-6$, x^3+4x^2+x-6, x^3-3x+2.

62. $x^3-7x^2+14x-8$, $x^3-6x^2+5x+12$, x^2-5x+4.

63. x^3+2x^2-x-2, x^4+x^3-x-1, and x^3-x^2.

64. x^4+xy^3, x^3y+y^4, and $x^4+x^2y^2+y^4$.

65. x^3+3x^2-x-3, x^5-x^2, and x^2-3x+2.

66. x^4-4x^2+3, $x^5-x^3+x^2-1$, x^3-2x^2-x+2.

67. x^3+2x^2-x-2, x^7+x^4, and x^2+4x+3.

68. x^2-2x-3, $x^2-7x+12$, and x^3+x^2-9x-9.

69. x^3+3x^2-x-3, x^4-2x^2-x+2, and x^5-x^4.

70. x^3+5x^2-x-5, x^5-x^2, and x^5-x^4+x-1.

EXERCISE 97.*

L. C. M. OF EXPRESSIONS THAT CAN BE FACTORED

209. The lowest common multiple of two or more expressions has already been defined (see Art. 86) and a

* The order of subjects given in this and the following six exercises is the one usually given in text books on Algebra, but some authors, notably Hall and Knight of England, prefer to take up the subjects after H.C.F. in the following order: I, Fractions Reduced to Lowest Terms; II, Multiplication of Fractions; III, Division of Fractions; IV, L. C. M.; V, Addition of Fractions; VI, Subtraction of Fractions. This order has the advantage of *using* the H.C.F. immediately after it is treated, and also using the L.C.M. immediately after it is treated. At the option of the teacher the order here spoken of may be substituted for the order given in the text.

method of finding the L. C. M. of monomials has been given. We now take up the case of the L. C. M. of polynomials. First we will consider the case of expressions which can be readily factored. The method of finding the L. C. M. in this case is exactly like that given for monomials, viz.:

Resolve each expression into its prime factors and form a product in which each of them occurs as many times as it occurs in that one of the given expressions in which it occurs the greatest number of times.

EXAMPLES.

Find the L. C. M. of the following expressions:

1. x^3-1 and x^2-1.
2. x^3-1, x^2-1, and $x-1$.
3. $x^2+7x+12$, $x^2+8x+15$, and x^2+5x+6.
4. x^2-x-6, x^2+x-2, and $x^2-4x-12$.
5. x^2+x-42, $x^2-11x+30$, and $x^2+2x-35$.
6. $4ab(a^2-3ab+2b^2)$ and $5a^2(a^2+ab-6b^2)$.
7. x^2y-xy^2, $3x(x-y)^2$, and $4y(x-y)^3$.
8. $x-y$, $x+y$, x^2-y^2, and x^4-y^4.
9. $a-b$, $a+b$, a^2-b^2, and a^2+b^2.
10. x^3-4x^2+3x, $x^4+x^3-12x^2$, $x^5+3x^4-4x^3$.
11. $20(x^2-1)$, $24(x^2-x-2)$, and $16(x^2+x-2)$.
12. x^2-7x+6, x^2-5x-6, and x^2-1.
13. $2(x-y)^2$, $8(x-y)$, $3(x+y)$, and $6(x^2+y^2)$.
14. x^2+7x+6, x^2+6x-7, and x^2-6x-7.
15. x^2-x-42, $x^2-9x+14$, and $x^2+4x-12$.

16. $5(a^2-2ab)$, $10(ab+2b^2)$, and $15(a^2b^2-4b^4)$.

17. x^2-5x+6, x^3-8, and x^5-27x^2.

18. x^2+x-2, x^3+8, and x^6-x^5.

19. $(2m+2)^2$, $(m+1)^3$, $5m+5$, and m^2-1.

20. $(x-1)^3$, $(2x-2)^2$, and x^4-2x^2+1.

EXERCISE 98

L. C. M. of Expressions Not Easily Factored.

210. When we wish to find the L. C. M. of two expressions not easily factored, we first find the H. C. F. of the two expressions by the method of exercise 96. This H. C. F. is of course *one* factor of each of the two given expressions, and the other factor is obtained by dividing each expression in turn by the H. C. F. In this way we *factor* each of the given expressions, and then, when the factors are known, we proceed as in the previous exercise to find the L. C. M. of the two expressions.

Let us find the L. C. M. of

$$6x^3-11x^2y+2y^3 \text{ and } 9x^3-22xy^2-8y^3.$$

We *first* find the H. C. F. of these two expressions as follows:

$$\begin{array}{rl}
 & 6x^3-11x^2y \qquad\qquad +2y^3 \\
 & 3 \\
\hline
9x^3-22xy^2-8y^3\ ) & 18x^3-33x^2y \qquad\qquad +6y^3\ (\ 2 \\
 & 18x^3 \qquad\qquad -44xy^2-16y^3 \\
\hline
-11y\ ) & -33x^2y+44xy^2+22y^3 \\
\hline
 & 3x^2-4xy-2y^2
\end{array}$$

$$\begin{array}{rl}
3x^2-4xy-2y^2\ ) & 9x^3 \qquad\qquad -22xy^2-8y^3\ (\ 3x+4y \\
 & 9x^3-12x^2y-6xy^2 \\
\hline
 & 12x^2y-16xy^2-8y^3 \\
 & 12x^2y-16xy^2-8y^3 \\
\hline
\end{array}$$

From this work we see that $3x^2-4xy-2y^2$ is the H. C. F. of the two given expressions, and if we divide each expression in turn by this H. C. F. we obtain the other factors as follows :

$$\begin{array}{l} 3x^2-4xy-2y^2 \;)\; 6x^3-11x^2y \qquad +2y^3 \;(\; 2x-y \\ \qquad\qquad \underline{6x^3-\;8x^2y-4xy^2} \\ \qquad\qquad\quad -\;3x^2y+4xy^2+2y^3 \\ \qquad\qquad\quad \underline{-\;3x^2y+4xy^2+2y^3} \end{array}$$

$$\begin{array}{l} 3x^2-4xy-2y^2 \;)\; 9x^3 \qquad -22xy^2-8y^3 \;(\; 3x+4y \\ \qquad\qquad \underline{9x^3-12x^2y-\;6xy^2} \\ \qquad\qquad\quad +12x^2y-16xy^2-8y^3 \\ \qquad\qquad\quad \underline{+12x^2y-26xy^2-8y^3} \end{array}$$

From the first of these two divisions we have

$$6x^3-11x^2y+2y^3=(3x^2-4xy-2y^2)(2x-y),$$

and from the second of these divisions we have

$$9x^3-22xy^2-8y^3=(3x^2-4xy-2y^2)(3x+4y).$$

Now we have the two given expressions factored, and from these factors we can readily write down the L. C. M. of the two given expressions. Plainly, this L. C. M. is

$$(3x^2-4xy-2y^2)(2x-y)(3x+4y).$$

Examples.

Find the L. C. M. of the following expressions :

1. $x^3+8x^2+19x+12$ and $x^3+7x^2+7x-15$.
2. $x^3+8x^2+19x+12$ and $x^3+3x^2-4x-12$.
3. $x^3+7x^2+7x-15$ and $x^3+3x^2-4x-12$.
4. $5x^2+11x+2$ and $15x^4+48x^3+9x^2$.
5. $4x^3-10x^2+4x+2$ and $3x^4-2x^3-3x+2$.

6. $6x^2+11xy+4y^2$ and $4x^2-8xy-5y^2$.

7. x^3+x^2-5x+3 and x^3-3x^2+3x-1.

8. $x^3+xy^2+2y^3$ and $x^3+x^2y+4y^3$.

9. $x^5+2x^4+x^3+8x^2+16x+8$ and $x^5-4x^3+x^2-4$.

10. $x^5-4x^3+x^2-4$ and $x^3+2x^2-5x-12$.

211. If we wish the L. C. M. of more than two expressions, we first find the L. C. M. of any two of them and then the L. C. M. of this result and the third expression, and so on until all the expressions are used; the result is the L. C. M. of all the given expressions.

EXAMPLES.

Find the L. C. M. of the following expressions:

11. $2x^2+2x-1$, $3x^3-4x+1$, and $2x^3-3x+1$.

12. x^3+2x^2+9, x^3-8x+3, and x^2-3x+1.

13. x^2-2x-2, x^3-4x^2+3, and x^3-3x^2+2.

14. $9x^3+2x+1$, $3x^3-8x^2+1$, and x^2-3x+1.

15. x^2-3x+2, $x^3-6x^2+11x-6$, and x^2-5x+6.

EXERCISE 99.

FRACTIONS REDUCED TO LOWEST TERMS.

212. The general properties of fractions have already been given, and all the operations connected with fractions have been given in the case that both numerator and denominator are *monomials*. It only remains now to apply the general properties and methods to fractions in which the numerator and denominator may be polynomials as well as monomials. The first thing we have

to consider is the reduction of fractions to their lowest terms. This is done exactly as before, by dividing both numerator and denominator by the H. C. F. of the numerator and denominator.

EXAMPLES.

Reduce the following fractions to their lowest terms:

1. $\frac{a^3+x^3}{a^4+a^2x^2+x^4}$.

2. $\frac{3x^3+2a(a+x)-a^3}{3x^3+4a(a+x)+a^3}$.

3. $\frac{2a^2+ab-b^2}{a^3+a^2b-a-b}$.

4. $\frac{x^2+ax+cx+ac}{x^2+bx+cx+bc}$.

5. $\frac{(3x-2y)^2-(2x+2y)^2}{2x-8y}$.

6. $\frac{x^3-39x+70}{x^2-3x-70}$.

7. $\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.

8. $\frac{3x^3-6x^2+x-2}{x^3-7x+6}$.

9. $\frac{x^3+x^2+x-3}{x^3-x-6}$.

10. $\frac{x^3+a^3}{x^2+2ax+a^2}$.

11. $\frac{a^3+2a^2-13a+10}{a^3+a^2-10a+8}$.

12. $\frac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$.

13. $\frac{4a^4+11a^2+25}{4a^4-9a^2+30a-25}$.

14. $\frac{5a^3+2a^2-15a-6}{7a^3-4a^2-21a+12}$.

15. $\frac{a^3+6a^2+11a+6}{a^3+5a^2+6a}$.

16. $\frac{36a^4-13a^2+1}{30a^3-19a^2+1}$.

17. $\frac{a^3-10a^2+26a-8}{a^3-9a^2+23a-12}$.

18. $\frac{8c^3-10c^2-16c-3}{6c^4-22c^3+31c^2-23c-7}$.

19. $\frac{2a^4-9a^3-14a+3}{3a^4-14a^3-9a+2}$.

20. $\frac{1+2a-3a^2}{1-3a-2a^2+4a^3}$.

EXERCISE 100.

ADDITION OF FRACTIONS.

213. In exercise 58 we considered the subject of the addition of fractions in which the numerator and denominator were each monomials. We now extend the subject there treated to the case in which the numerator or denominator or both are polynomials. Of course the method is the same here as in exercise 58, viz.: *first, reduce the fractions to a common denominator, and then add the numerators.* Here, as before, we take the L. C. M. of the given denominators for a common denominator.

EXAMPLES.

Add the following fractions:

1. $\frac{4x-5}{6}$ and $\frac{3x-2}{12}$.

2. $\frac{4a-b}{4x^2}$ and $\frac{5a-b}{8x^3}$.

3. $\frac{4}{x-1}$ and $\frac{7}{x^2-1}$.

4. $\frac{n^3}{(n+r)^2}$ and $\frac{nr}{(n-r)^2}$.

5. $\frac{3a-4}{a+b}$, $\frac{5a-6}{a-b}$, and $\frac{7a-3}{a^2-b^2}$.

6. $\frac{3}{x^2+2x}$ and $\frac{7}{x^2-2x}$.

7. $\frac{x+y}{x-y}$, $\frac{x-y}{x+y}$, and $\frac{2x+3y}{x^2+y^2}$.

8. $\frac{u+v+w}{u+v}$, $\frac{u+v+w}{u-v}$, and $\frac{1}{u^2-v^2}$.

9. $\frac{n}{n+r}$, $\frac{r}{n-r}$, and $\frac{1}{n^2+r^2}$.

10. $\frac{a+b}{a^2+b^2}$, $\frac{a-b}{a^2-b^2}$, and $\frac{ab}{a^3+b^3}$.

11. $\frac{a+b}{x(x-a)}$ and $\frac{a+b}{x(x-b)}$.

12. $\frac{x^2+y^2}{x^3-y^3}$, $\frac{x+y}{x^2-y^2}$, and $\frac{1}{x-y}$.

13. $\frac{1}{a+b+c}$, $\frac{2}{a+b}$, and $\frac{3}{a+c}$.

14. $\frac{x+1}{x^2+5x+6}$, $\frac{2x+3}{(x+2)^2}$, and $\frac{3x+4}{(x+3)^2}$.

15. $\frac{2x+5}{x^2+5x+6}$ and $\frac{2x+7}{x^2+7x+10}$.

16. $\frac{x}{x^3-y^3}$, $\frac{y}{x^2+xy+y^2}$, and $\frac{1}{x-y}$.

17. $\frac{x}{x+a}$, $\frac{x}{x+b}$ and $\frac{x^2+c^2}{x^2+ax+ac}$.

18. $\frac{x^2+a^2}{x^4+a^2x^2+a^4}$, $\frac{x+a}{x^2+ax+a^2}$, and $\frac{x-a}{x^2-ax+a^2}$.

19. $\frac{a^2-bc}{bc}$, $\frac{b^2-ac}{ac}$, and $\frac{c^2-ab}{ab}$.

20. $\frac{1}{x^2-3x+2}$, $\frac{1}{x^2-x-2}$, and $\frac{1}{x^2-1}$.

EXERCISE 101.

SUBTRACTION OF FRACTIONS.

214. In exercise 59 we considered the subject of subtraction of fractions in which both numerator and denominator were monomials, and it only remains now to take up fractions in which numerator or denominator or both are polynomials. The method is of course the same as in exercise 59, viz.: *reduce the fractions to a common denominator and then subtract the numerator of the subtrahend from that of the minuend.*

EXAMPLES.

1. From $\frac{2x+1}{3}$ take $\frac{3x+2}{4}$.

2. From $\frac{ax+b}{c}$ take $\frac{cx+b}{a}$.

3. From $\frac{x+y}{x-y}$ take $\frac{x-y}{x+y}$.

4. From $\frac{x^2+y^2}{x^3-y^3}$ take $\frac{x+1}{x^2-y^2}$.

5. From $\frac{x^2+y^2}{x^3-y^3}$ take $\frac{x+y}{x^2+xy+y^2}$.

6. From $\frac{x+5}{x^2+5x+6}$ take $\frac{x+3}{x^2+7x+10}$.

7. From $\frac{xy}{x^3+y^3}$ take $\frac{x^2y+xy^2}{x^4-y^4}$.

8. From $\frac{x^2+xy+y^2}{x^2-y^2}$ take $\frac{x^2-y^2}{x^2+xy+y^2}$.

9. From $\frac{x^2+xy+y^2}{x+y}$ take $\frac{x^2-xy+y^2}{x-y}$.

10. From $\frac{a}{(x-a)(a-b)}$ take $\frac{b}{(x-b)(a-b)}$.

11. From $\frac{b}{x^2+(a+b)x+ab}$ take $\frac{c}{x^2+(a+c)x+ac}$.

In the following examples, perform the additions and subtractions indicated and express the result as a single fraction in its lowest terms:

12. $\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}$.

13. $\frac{x+1}{(x+2)(x+3)}+\frac{x+2}{(x+1)(x+3)}-\frac{x+3}{(x+1)(x+2)}$.

14. $\frac{1}{x^2-3x+2}+\frac{2}{x^2-x-2}-\frac{3}{x^2-1}$.

15. $\left(\frac{a}{2}+\frac{b}{3}-\frac{c}{4}\right)+\left(\frac{a}{4}-\frac{3b}{4}-\frac{5c}{6}\right)$.

16. $\frac{a+b}{a-b}+\frac{b-a}{a+b}-\frac{4ab}{a^2-b^2}$.

17. $\frac{2x+1}{(x-1)(x-2)}-\frac{3x+2}{(x-2)(x-3)}-\frac{4x+3}{(x-3)(x-4)}$.

18. $\frac{a}{a-1}+\frac{a+1}{a^2+1}-\frac{a}{a+1}-\frac{a^2-a+1}{a^2-1}$.

19. $\frac{3a-6}{a^2-7a+12}-\frac{4a-5}{a^2-8a-15}-\frac{a-1}{a^2-9a+20}$.

20. $\frac{2a+1}{(a^2+1)^2}+\frac{3a+2}{a^2+1}-\frac{2}{a-1}+\frac{3}{a+2}$.

EXERCISE 102.

MULTIPLICATION OF FRACTIONS.

215. In exercise 60 we considered the multiplication of fractions in which both numerator and denominator were monomials, and it only remains here to extend the operation to the case where the numerator or denominator or both are polynomials. Of course the method used here is the same as in exercise 60, viz.: *multiply all the numerators together for a new numerator and all the denominators together for a new denominator.* The result should be reduced to its lowest terms if it is not in its lowest terms already.

216. Instead of actually performing the multiplications it will frequently be best to *indicate* them by using parentheses, for sometimes in the result the numerator and

denominator will contain a common factor, which can be struck out and thus save the trouble of multiplying by these factors. For example, if we wish the product of

$$\frac{a+x}{b}, \quad \frac{b}{c+x}, \text{ and } \frac{c-x}{a+x},$$

we can write the product thus:

$$\frac{b(a+x)(c-x)}{b(c+x)(a+x)}.$$

Here the numerator and denominator contain the common factor $b(a+x)$, which being rejected from both numerator and denominator leaves the result simply

$$\frac{c-x}{c+x}.$$

EXAMPLES.

Find the product of the following fractions:

1. $\frac{b+c}{x+c}$ and $\frac{b-c}{x-c}$.

2. $\frac{x+1}{y+1}$ and $\frac{x^2-x+1}{y^2-y+1}$.

3. $\frac{a-b}{c-d}$ and $\frac{a^2+ab+b^2}{c^2+cd+d^2}$.

4. $\frac{a^2b^2+3ab}{4a^2-1}$ and $\frac{2a+1}{ab+3}$.

5. $\frac{x^2+xy+y^2}{x^2-y^2}$ and $\frac{x^2-xy+y^2}{x^2+y^2}$.

6. $\frac{a^2-121}{a^2-4}$ and $\frac{a+2}{a+11}$.

7. $\frac{2x^2-13x+15}{4x^2-9}$ and $\frac{4x^2-1}{2x^2+11x+5}$.

8. $\frac{14x^2-7x}{12x^3+24x^2}$ and $\frac{x^2+2x}{2x-1}$.

9. $\frac{3a^2-4ab}{7(a+b)}$, $\frac{14(a-b)}{3a^2+3b^2}$, and $\frac{a+b}{a-b}$.

10. $\frac{a^2-b^2}{a^2-2ab+b^2}$, $\frac{a^2-b^2}{a}$, and $\frac{1}{a+b}$.

11. $\frac{1-a^2}{b+b^2}$, $\frac{1-b^2}{a+a^2}$, and $\frac{b}{1-a}$.

12. $\frac{1+a^2}{1+b}$, $\frac{1+b^2}{a-a^2}$, and $\frac{1}{1-a}$.

13. $\frac{a^3-x^3}{a^3+x^3}$ and $\frac{a^2-x^2}{a^2-2ax+x^2}$.

14. $\frac{(a+b)^2}{2ab}$, $\frac{ab^2}{a^3+b^3}$, and $\frac{a^2-ab+b^2}{4a(a+b)}$.

15. $\frac{a^2-4b^2}{a(a+2b)}$ and $\frac{ab+4b^2}{a(a-2b)}$.

16. $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}$ and $\frac{a^2+x^2}{a^2-x^2}$.

17. $\frac{a^3+3a^2b+3ab^2+b^3}{a^2-b^2}$, $\frac{a-b}{a^2+ab}$, and $\frac{a}{a+b}$.

18. $\frac{x^2-9x+20}{x^2-6x}$, $\frac{x^2-13x+42}{x^2-5x}$, and $\frac{x^2}{x-7}$.

19. $\frac{x^2-xy+y^2}{x^2-y^2}$, $\frac{x+y}{x-y}$, and $\frac{(x-y)^2}{x^3+y^3}$.

20. $\frac{(x-3)^2}{x-1}$, $\frac{x^2-2x+1}{x^2-9}$, and $\frac{x+3}{x^2-4x+3}$.

EXERCISE 103.

DIVISION OF FRACTIONS.

217. In exercise 61 we considered the subject of the division of fractions in which both numerator and denominator were monomials, and we now have to extend the subject to the case where the numerator or denominator or both are polynomials. The method used here is of course the same as in exercise 61, viz.: *invert the terms of the divisor and proceed as in multiplication.*

EXAMPLES.

1. Divide $\frac{(a+x)^2}{a-x}$ by $\frac{a+x}{x}$.

2. Divide $\frac{x^2-13x+42}{x^2-5x}$ by $\frac{x-7}{x^2}$.

3. Divide $\frac{x^2-9x+20}{x^2-6x}$ by $\frac{x^2-5x}{x^2-13x+42}$.

4. Divide $\frac{a^3+x^3}{a^3-x^3}$ by $\frac{a^2-x^2}{a^2-2ax+x^2}$.

5. Divide $\frac{x^2+9x+20}{x^2+7x+10}$ by $\frac{x^2+7x+12}{x^2+6x+8}$.

6. Divide $\frac{x^2+6x+8}{x^2+9x+20}$ by $\frac{x^2-4}{x^2-16}$.

7. Divide $\frac{a^2-3a-4}{a^2-1}$ by $\frac{a^2-16}{a^2-a}$.

8. Divide $\frac{x^3+3ax^2+3a^2x+a^3}{x^2+2bx+b^2}$ by $\frac{x^2-a^2}{x^2-bx}$.

9. Divide $\frac{x^2+2xy+y^2-z^2}{2xy}$ by $\frac{2yz+y^2+z^2-x^2}{2yz}$

10. Divide $\frac{x^2+2xy+y^2-z^2}{x+y-z}$ by $\frac{x-y-z}{x^2-2xy+y^2-z^2}$.

11. Divide $\frac{(x+1)^2}{x^2+1}$ by $\frac{x^2-1}{(x-1)^2}$.

12. Divide $\frac{(x+1)^2}{x^2+1}$ by $\frac{(x-1)^2}{x^2-1}$.

13. Divide $\frac{x^3+y^3}{x^3-z^3}$ by $\frac{x+y}{x-z}$.

14. Divide $\frac{a^3+3a^2+3a+1-b^3}{a^2+2a+1-b^2}$ by $\frac{a+b+1}{ab+1}$.

15. Divide $\frac{1}{1-a^2}$ by $\frac{1-a}{(1+a)^2}$.

16. Divide $\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2}$ by $\frac{a+b+c}{a+b}$.

17. Divide $\frac{a^3-b^3+a-b}{a^2-b^2+a+b}$ by $\frac{a-b}{a+b}$.

18. Divide $\frac{(x+y)^2-z^2}{(x-y)^2-z^2}$ by $\frac{x+y-z}{x-y+z}$.

19. Divide $\frac{a^4+a^2b^2+b^4}{a^4-b^4}$ by $\frac{a^2-ab+b^2}{a^2-b^2}$.

20. Divide $\frac{a^2x^2-1}{x-1}$ by $\frac{1}{1+ax}$.

EXERCISE 104.

MISCELLANEOUS FRACTIONS.

218. We may take an integral expression (*i. e.*, a monomial or polynomial in which there is no fraction involved,) along with one or more fractions, giving us a form partly integral and partly fractional. Such expressions are sometimes called **Mixed Expressions** or numbers. Mixed expressions may always be reduced to the form of fractions by writing the integral part in the form of a fraction with a denominator 1, and then performing the indicated operations as before explained. For example, suppose we wish to express

$$x^2+y^2+\frac{xy}{x^2-y^2}$$

in the form of a fraction. We write the expression thus:

$$\frac{x^2+y^2}{1}+\frac{xy}{x^2-y^2}.$$

Reducing to a common denominator and adding in the usual way, we get

$$\frac{x^4-y^4+xy}{x^2-y^2}.$$

This process may be called **Reducing Mixed Expressions to Fractions.**

EXAMPLES.

Reduce the following mixed expressions to fractions:

1. $x+\frac{1}{x}$.
2. $x^2+1+\frac{2}{x}$.
3. $x+a-\frac{3}{x-a}$.
4. $x+y+\frac{1}{x}+\frac{1}{y}$.
5. $a+b+c-\frac{b}{a}-\frac{c}{b}-\frac{a}{c}$.
6. $x^2+x+1-\frac{x}{x+1}$.
7. $a+b+c+\frac{1}{a+b+c}$.
8. $x^2-ax+a^2+\frac{b^3}{x-a}$.
9. $(x+a)^2+\frac{1}{x+a}$.
10. $ax+by-\frac{1}{ax-by}$.

219. When the given mixed expression has only one fraction, as in the example $x^2+y^2+\frac{xy}{x^2-y^2}$, we reduced to a common denominator by multiplying numerator and denominator of $\frac{x^2+y^2}{1}$ by x^2-y^2, the denominator of the fractional part of the given expression. This amounts to multiplying the integral part by the given denominator and adding the numerator exactly as in Arithmetic.

Since the fraction was produced from the mixed expression by multiplying the denominator by the integral part and adding the numerator, it follows that to reverse this process and go back from the fraction to the mixed

expression we would *divide the numerator by the denominator as far as possible and write the quotient for the integral part and the remainder over the denominator for the fractional part*, which again is exactly as in Arithmetic.

EXAMPLES.

Change the following fractions to mixed expressions:

11. $\dfrac{x^3-x^2y+xy^2-y^3}{x+y}$.

12. $\dfrac{x^2+ax+a^2}{x^2+a}$.

13. $\dfrac{30x^2-4x+9}{10}$.

14. $\dfrac{12a^2+4a-5c}{4a}$.

15. $\dfrac{x^4+a^2x^2+a^4+x+a}{x^2+ax+a^2}$.

16. $\dfrac{x^3+3ax^2+3a^2x+a^3+a^2}{x+a}$.

17. $\dfrac{x^2+y^2+z^2}{x+y}$.

18. $\dfrac{x^2-y^2+z^2}{x+y}$.

19. $\dfrac{x^6-y^6-z^6}{x^2-y^2}$.

20. $\dfrac{ax+by+c}{x+y}$.

220. By combinations or repeated applications of the preceding processes we are able to deal with more complicated cases than have yet been given. For example, let us take the fraction

$$\frac{\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a}+\dfrac{x-a}{x+a}}.$$

$$\text{The numerator} = \frac{x+a}{x-a}-\frac{x-a}{x+a} = \frac{(x+a)^2-(x-a)^2}{x^2-a^2},$$

$$= \frac{(x^2+2ax+a^2)-(x^2-2ax+a^2)}{x^2-a^2},$$

$$= \frac{4ax}{x^2-a^2}.$$

$$\text{The denominator} = \frac{x+a}{x-a}+\frac{x-a}{x+a},$$

$$= \frac{(x+a)^2+(x-a)^2}{x^2-a^2},$$

$$= \frac{(x^2+2ax+a^2)+(x^2-2ax+a^2)}{x^2-a^2},$$

$$= \frac{2(x^2+a^2)}{x^2-a^2}.$$

Therefore the original fraction

$$= \frac{4ax}{x^2-a^2} \div \frac{2(x^2-a^2)}{x^2-a^2}.$$

$$= \frac{4ax}{x^2-a^2} \times \frac{x^2-a^2}{2(x^2+a^2)},$$

$$= \frac{4ax}{2(x^2+a^2)} = \frac{2ax}{x^2+a^2}.$$

This final result is much simpler than the fraction we started with, so this kind of work may be called **Simplifying Complex Fractions or Expressions Involving Fractions.**

EXAMPLES.

Simplify the following expressions:

21. $\dfrac{\frac{x^2}{3}+x+3}{x+3+\frac{9}{x}}$.

22. $\dfrac{\frac{x^2+y^2}{y}-x}{\frac{1}{y}-\frac{1}{x}} \times \dfrac{x^2-y^2}{x^3+y^3}$.

23. $\dfrac{x-2}{x+\frac{1}{x-1}}+\dfrac{2x^2+x-1}{x^3+1}$.

24. $\dfrac{\frac{a}{b} \div \frac{a+b}{b}}{a^2-ab+b^2-\frac{2b^3}{a+b}}$.

25. $\left(\dfrac{1}{x}+\dfrac{1}{2x}\right) \div \left(\dfrac{1}{x^2}-\dfrac{1}{2x^2}\right)$

26. $\dfrac{1}{1-\frac{1}{1+x}}+\dfrac{1}{1-\frac{1}{1-x}}$.

27. $\dfrac{(a^2x+b^2y)(x+y)-(ax+by)^2}{(a-b)^2}$.

28. $\left(x^2-x+\dfrac{x-1}{x+1}\right)\div\left(x+\dfrac{1}{x+1}\right)$.

29. $\left(1-\dfrac{b^2}{a^2}\right)\times\left(1-\dfrac{ab-b^2}{a^2}\right)\times\dfrac{a^4}{a^3+b^3}\times\dfrac{a-b}{a^2+b^2}$.

30. $\dfrac{a^3+b^3}{a^2-2ab+b^2}\times\dfrac{a^2+ab+b^2}{a^3+b^3}\div\dfrac{a^2+ab+b^2}{a^3-3ab(a-b)-b^3}$.

31. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}\times\dfrac{(a-b)^2-(c-d)^2}{(a-c)^2-(b-d)^2}$.

32. $\left(\dfrac{n}{n-1}+\dfrac{n}{n+1}\right)\div\left(\dfrac{n}{n-1}-\dfrac{n}{n+1}\right)$.

33. $\left(\dfrac{1}{n-r}-\dfrac{1}{n-s}\right)\div\dfrac{r^2}{n^2-n(r+s)+rs}$.

34. $\left(\dfrac{x+y}{y}+\dfrac{2(y-x)}{x-y}\right)\div\left(\dfrac{1}{x}-\dfrac{1}{y}\right)$.

35. $\left(\dfrac{a^3-b^3}{a-b}-\dfrac{a^3+b^3}{a+b}\right)\div\dfrac{4ab}{a^2-b^2}$.

36. $\left[\left(\dfrac{a}{b}+2\right)^2-\left(\dfrac{a}{b}-2\right)^2\right]\div\left[\left(\dfrac{a}{b}+2\right)^2+\left(\dfrac{a}{b}-2\right)^2\right]$

37. $\left(\dfrac{x^4-a^4}{x^2-2ax+a^2}\div\dfrac{x^2+ax}{x-a}\right)\times\dfrac{x^5-a^2x^3}{x^3+a^3}\div\left(\dfrac{x}{a}-\dfrac{a}{x}\right)$.

38. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}-2}{a-b}$.

39. $\dfrac{1-\dfrac{1}{x}}{\dfrac{1}{x^2}}$.

40. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}$.

41. $\dfrac{\dfrac{1}{x}}{\dfrac{1}{x}-\dfrac{1}{y}}\div\dfrac{(x-y)^2-4xy}{\dfrac{1}{x^2}+\dfrac{1}{y^2}-\dfrac{2}{xy}}$.

42. $\dfrac{(ay-bx)^2+(ax+by)^2}{\left(\frac{a}{b}+\frac{b}{a}\right)\left(\frac{x}{y}+\frac{y}{x}\right)}$.

43. $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$.

Fractions like this are called **Continued Fractions.** To simplify a continued fraction, begin at the lowest part and proceed upward step by step as follows:

$$1+\frac{x}{4-x}=\frac{4-x+x}{4-x}=\frac{4}{4-x}.$$

Hence the original fraction may be written

$$\frac{1}{x-1+\dfrac{1}{\dfrac{4}{4-x}}}$$

But $$\frac{1}{\dfrac{4}{4-x}}=\frac{4-x}{4}.$$

Hence the original fraction may be written

$$\frac{1}{x-1+\dfrac{4-x}{4}}$$

But $$x-1+\frac{4-x}{4}=\frac{4x-4+4-x}{4}=\frac{3x}{4}.$$

Hence the original fraction $=\dfrac{1}{\frac{3x}{4}}=\dfrac{4}{3x}$.

44. $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$.

45. $\dfrac{\dfrac{x}{a}}{\dfrac{x}{a}+\dfrac{a}{x-\dfrac{a^2}{x}}}$.

46. $\dfrac{x^2}{a+\dfrac{x^2}{a+\dfrac{x^2}{a}}}$.

47. $\left[x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)\right]\div\left(x-\dfrac{1}{x}\right)$.

48. $\dfrac{\dfrac{a}{b}+\dfrac{b}{a}-1}{\dfrac{a^2}{b^2}+\dfrac{a}{b}+1} \times \dfrac{1+\dfrac{b}{a}}{a-b} \div \dfrac{1+\dfrac{b^3}{a^3}}{\dfrac{a^2}{b}-\dfrac{b^2}{a}}$.

49. $\dfrac{\dfrac{x-1}{3}+\dfrac{x-1}{x-2}}{\dfrac{x+2}{x}+\dfrac{x+2}{x-3}} \div \dfrac{\dfrac{x+3}{7}-\dfrac{x+3}{x+4}}{\dfrac{x-2}{3}+\dfrac{x-2}{x-1}}$.

50. $\dfrac{1+\dfrac{1}{x}}{x^2+\dfrac{1}{x}} \div \dfrac{\dfrac{1}{x}}{x-1+\dfrac{1}{x}}$.

221. When an operation is performed upon a polynomial, some or all of whose terms are fractions, we naturally combine all the terms into a single fraction, and *then* perform the indicated operation. Sometimes, however, the operation may be performed without thus combining. For instance, if we wish to multiply

$$\frac{a^2}{2}+\frac{a}{3}+\frac{1}{4} \text{ by } \frac{a}{2}-\frac{1}{3},$$

we would by the previous process combine each of these expressions into a single fraction as follows:

$$\frac{a^2}{2}+\frac{a}{3}+\frac{1}{4}=\frac{6a^2+4a+3}{12}$$

and

$$\frac{a}{2}-\frac{1}{3}=\frac{3a-2}{6}.$$

Therefore,

$$\left(\frac{a^2}{2}+\frac{a}{3}+\frac{1}{4}\right)\left(\frac{a}{2}-\frac{1}{3}\right)=\frac{6a^2+4a+3}{12}\times\frac{3a-2}{6},$$

$$=\frac{18a^3+a-6}{72}.$$

But we may also multiply these expressions together the same as integral expressions were multiplied, as follows:

$$\begin{array}{l} \frac{a^2}{2}+\frac{a}{3}+\frac{1}{4} \\ \frac{a}{2}-\frac{1}{3} \\ \hline \frac{a^3}{4}+\frac{a^2}{6}+\frac{a}{8} \\ \quad -\frac{a^2}{6}-\frac{a}{9}-\frac{1}{12} \\ \hline \frac{a^3}{4}+\frac{a}{8}-\frac{a}{9}-\frac{1}{12} \end{array}$$

The terms of this result may be combined after reducing to a common denominator, 72. Combining terms,

$$\frac{a^3}{4}+\frac{a}{8}-\frac{a}{9}-\frac{1}{12}=\frac{18a^3+9a-8a-6}{72}=\frac{18a^3+a-6}{72},$$

which is the same as was reached by the other process.

222. As another illustration let us divide

$$\frac{a^3}{6}-\frac{13a^2}{36}+\frac{1}{4} \text{ by } \frac{a}{3}-\frac{1}{2}$$

The work may be arranged as follows:

$$\begin{array}{l} \frac{a}{3}-\frac{1}{2}\Big)\frac{a^3}{6}-\frac{13a^2}{36}+\frac{1}{4}\Big(\frac{a^2}{2}-\frac{a}{3}-\frac{1}{2} \\ \quad \frac{a^3}{6}-\frac{a^3}{4} \\ \quad \hline \quad -\frac{a^2}{9}+\frac{1}{4} \\ \quad -\frac{a^2}{9}+\frac{a}{6} \\ \quad \hline \quad -\frac{a}{6}+\frac{1}{4} \\ \quad -\frac{a}{6}+\frac{1}{4} \\ \quad \hline \end{array}$$

The terms of this quotient may be combined, giving

$$\frac{3a^2-2a-3}{6},$$

which is the same result as would be obtained by reducing both dividend and divisor to a single fraction and proceeding by the method already given for dividing one fraction by another.

EXAMPLES.

By the method used in these two illustrations work the following examples:

51. Multiply $x^2+\frac{1}{x}$ by $x-\frac{1}{x}$.

52. Multiply $\frac{x^2}{2}+\frac{x}{3}+\frac{1}{2}$ by $\frac{x}{3}+\frac{1}{4}$.

53. Multiply $\frac{x^3}{a}+\frac{x^2}{b}-\frac{1}{ab}$ by $\frac{x^2}{a}-\frac{x}{b}$.

54. Divide $\frac{x^3}{6}-\frac{17x^2}{72}+\frac{x}{12}+1-\frac{2}{3x}$ by $\frac{x}{2}-\frac{1}{3}$.

55. Divide $\left(\frac{1}{a}+\frac{1}{b}\right)^2-1$ by $\frac{1}{a}+\frac{1}{b}-1$.

56. Divide $\frac{a^4}{16}-\frac{1}{a^2}$ by $\frac{a^2}{4}-\frac{1}{a}$.

57. Divide $a^3-\frac{a^2}{3}+a-\frac{4}{3}-\frac{1}{a^2}$ by $a+\frac{1}{3}$.

58. Divide $8a^3+\frac{27}{a^3}$ by $2a+\frac{3}{a}$ and multiply the result by $a+\frac{1}{a}$.

59. Multiply $x^2-\frac{1}{x^2}$ by $x+\frac{1}{x}$ and divide the result by $x-\frac{1}{x}$.

CHAPTER XIV.

QUADRATIC EQUATIONS.

EXERCISE 105.

PRELIMINARY TOPICS.

223. The **Degree of a Monomial** *with respect to any letter or letters it may contain* is the sum of the exponents of the letters named; unity being always understood where no exponent is written. Thus, $5ab^2x^3y^4$ is of the first degree with respect to a, of the second degree with respect to b, of the third degree with respect to x, of the fourth degree with respect to y, of the seventh degree with respect to x and y, of the seventh degree with respect to a, b, and y, etc.

1. What is the degree of $2a^2b^3x^4y^5$ with respect to a? What with respect to x? What with respect to y? What with respect to a, b and x? What with respect to x and y?

2. What is the degree of $3a^2x^2y^3$ with respect to x? What with respect to y? What with respect to x and y? What with respect to a and x? What with respect to a, x and y?

224. When the degree of a monomial is spoken of without specifying the letters with respect to which the degree is taken, it is usually understood to mean the degree with respect to *all* the letters it contains, and is then equal to the number of literal prime factors, or what is the same thing, the sum of all the exponents of the letters in the expression.

225. The **Degree of a Polynomial** *with respect to any letter or letters it may contain* is the degree of that one of its terms whose degree with respect to the specified letters is highest. Thus,

$$a^2x^3+abc^4x+e^2x^4y^5$$

is of the second degree with respect to a, because the first term is of the second degree respect to a, and neither of the other terms are of so high degree with respect to a.

The same expression is of the fourth degree with respect to x, because the third term is of the fourth degree with respect to x and neither of the other terms are of so high degree with respect to x.

3. What is the degree $ax^2y+bxy^2+x^2y^2$ with respect to x? What with respect to y? What with respect to x and y? What with respect to a? What with respect to a and b?

4. What is the degree of $x^2y+xy^3+x^5$ with respect to x? What with respect to y? What with respect to x and y?

Ans. 5, for the degree with respect to x alone is 5, and the term that determines the degree has *no* y, so it leaves the degree 5.

5. What is the degree of $x^3ax^2y+bxy^2+aby^3$ with respect to x? What with respect to y? What with respect to x and y?

6. What is the degree of $a^2bx+b^2xy^2+cxy^3$ with respect to a? What with respect to x? What with respect to y? What with respect a and x? What with respect to a, b, c, x and y.

226. When the degree of a polynomial is spoken of without specifying the letters with respect to which the degree is taken, it is usually understood to mean the degree

with respect to *all* the letters it contains, and is then equal to the number of literal prime factors in that term which contains the greatest number of such literal prime factors.

227. The **Degree of an Equation** is its degree *with respect to the unknown numbers or quantities*, *i. e.*, it is the degree of that one of its terms whose degree with respect to the unknown number is the highest.

Remember that the last letters of the alphabet are used to stand for unknown numbers.

7. What is the degree of $a^3x+b^3y^2=4$? Of $ax^2+bxy+cy^2=d$? Of $ax^2+bx+c=0$?

228. A **Quadratic Equation** is only another name for an equation of the second degree. In this chapter we deal only with equations with *one* unknown number.

229. Quadratic Equations are divided into two classes: **Pure** or **Incomplete** and **Affected** or **Complete.**

230. A **Pure** or **Incomplete** quadratic equation is one which contains the second but not the first power of the unknown number, as $3x^2=12$ and $\frac{x^2+1}{5}=2$.

231. An **Affected** or **Complete** quadratic equation is one which contains both the second and first powers of the unknown number, as $3x^2+4x=39$ and $\frac{x^2}{8}+\frac{x}{4}=3$.

232. A **Root** of an equation is any number which substituted for the unknown number will satisfy the equation, *i. e.*, will cause the equation to be true. For example, 2 is a root of the equation $x^2+x=6$, for if 2 be

written in place of x we get $2^2+2=6$, or $4+2=6$, which is true. Again, -3 is also a root of $x^2+x=6$, for if -3 be written in place of x we get $(-3)^2-3=6$, or $9-3=6$, which is true.

Although we deal in this chapter only with equations of the second degree, still this definition of root will hold good for an equation of any degree whatever, but it must be understood that the word can be used only with reference to an equation of *one* unknown number.

The student must not confuse the word root as here used with the square root or cube root or some other root of expressions. See Art. 151.

233. The **Solution** of an equation is the process by which the roots are found.

EXERCISE 106.

PURE QUADRATIC EQUATIONS.

234. If $x^2=4$ we know that x must be some number which raised to the second power will give 4. Now, there are two such numbers, $+2$ and -2. Therefore, $x=2$ or -2. Either of these numbers will satisfy the given equation, *i. e.*, will render the given equation true. Also, if $x^2=9$ we know that x is a number which raised to the second power will give 9. Either $+3$ or -3 will satisfy the given equation. Therefore, $x=3$ or -3. So whatever number is placed equal to x^2 there are two numbers of opposite signs, but otherwise alike, which will satisfy the equation, or in other words, there are two values of x of opposite signs but otherwise just alike.

235. To solve a pure quadratic equation we reduce it so that all the unknown terms are on one side of the equation and all the known terms on the other side, then

we divide by the coefficient of the square of the unknown quantity, and lastly extract the square root of each side of the equation.

236. In solving a pure quadratic equation we usually write both values of the unknown quantity at once, as in the equation $x^2=4$ after extracting the square root of each side we would write $x=\pm 2$, using the double sign $\pm$ to show that either $+2$ or -2 will satisfy the given equation $x^2=4$.

The student may think that we should write the double sign on *both* sides of the equation instead of on one side only, thus, $\pm x=\pm 2$; but this would evidently mean $x=2$ or $x=-2$ or $-x=2$ or $-x=-2$. The third of these equations is really the same as the second and the fourth is really the same as the first, so that we really get no more values by writing $\pm x=\pm 2$ than we do by writing $x=\pm 2$.

237. Whenever we extract the square root of each side of an equation we should write the double sign $\pm$ on *one* side of the equation obtained.

EXAMPLES.

Solve the following equations:

1. $x^2+3=4$.
2. $x^2+3=7$.
3. $(x^2+1)+(x^2+2)+(x^2+3)=306$.
4. $(x^2-4)+(x^2+2)=x^2+14$.
5. $2(x^2+1)+3(x^2+\frac{3}{4})=5\frac{1}{2}$.
6. $3(x^2-1)+4(x^2-2)=5x^2+39$.
7. $3x^2-4=28+x^2$.
8. $5x^2-7=293+2x^2$.
9. $(x^2-1)-(x^2-2)-(x^2-3)=54-3x^2$.

10. $\frac{x^2-1}{5}+\frac{x^2+4}{10}+\frac{x^2-4}{3}=25-x^2.$

11. $\frac{2}{x^2+1}+\frac{4}{5}=1.$

13. $\frac{x}{4}=\frac{4}{x}.$

12. $\frac{x^2+1}{x^2-1}+\frac{1}{4}=\frac{3}{2}.$

14. $9x^2-16=0.$

15. $12x^2-75=0.$

16. $2(x^2-1)-3(x^2+1)+9=0.$

17. $\frac{x^2+1}{2}-\frac{x^2-1}{8}-10=0.$

18. $2x+\frac{3}{x}-\frac{5}{x}=8x+\frac{8}{x}.$

19. $(2x+1)^2=4x+2.$

20. $3x^2-199=(x+1)^2-2x.$

238. The equation $x^2=4$ may be written in the form $x^2-4=0$. Now, as the first member of this equation is the difference of two squares, it may be factored, and hence the equation may be written $(x-2)(x+2)=0$.

239. The product of two or more factors is equal to zero whenever any *one* of the factors is zero. Therefore the equation $(x-2)(x+2)=0$ can be satisfied in either of two ways: first, when $x-2=0$, *i. e.*, when $x=2$, and second, when $x+2=0$, *i. e.*, when $x=-2$.

As another example take the equation

$$5x^2-9=2x^2+18.$$

Transpose all the terms to the first member and we get

$$5x^2-2x^2-9-18=0,$$

or $$3x^2-27=0.$$

Dividing by 3, $$x^2-9=0.$$

Factoring, $$(x-3)(x+3)=0.$$

This equation can be satisfied in either of two ways: first, when $x-3=0$, *i. e.*, when $x=3$, and second, when $x+3=0$, *i. e.*, when $x=-3$.

240. By the method illustrated in these two examples we get the same roots as would be obtained by the former method. Thus we have another method of solving a pure quadratic equation, viz.: *Collect together into one term all the unknown numbers and into another term all the known numbers; write these two terms on the same side of the equation, making the other side zero; divide both members by the coefficient of the square of the unknown number (remembering that when zero is divided by anything the quotient is still zero); factor the resulting first member; put each factor separately equal to zero, and solve the resulting simple equations.*

EXAMPLES.

Solve the following equations by the method just explained:

21. $x^2-100=0$.
22. $4x^2-100=0$.
23. $5x^2=80$.
24. $x+\frac{1}{x}=\frac{26}{x}$.
25. $(x+2)^2=4(x+5)$.
26. $(x+2)(x+3)=5x+42$.
27. $x^2+x+1=x+101$.
28. $x^2-2x-3=33-2x$.
29. $(x+a+b)^2=2(a+b)x+2(a^2+b^2)$.
30. $(2x+1)^2=4x+82$.

EXERCISE 107.

AFFECTED QUADRATICS.

241. If we have given the equation $x^2=25$ we solve it by the preceding exercise, and find $x=\pm 5$. Similarly, if we have the equation $(x+1)^2=25$, we find $x+1=\pm 5$.

If we take the upper sign we get $x+1=5$, or $x=4$, and if we take the lower sign we get $x+1=-5$, or $x=-6$.

Thus we find the roots of the equation $(x+1)^2=25$, or what is the same, $x^2+2x+1=25$, or $x^2+2x=24$.

242. Therefore, to solve $x^2+2x=24$, we first add 1 to each member to make the first member a perfect square, and get $x^2+2x+1=25$; then we take the square root of each member, and get $x+1=\pm 5$, whence $x=4$ or -6.

243. Similarly, to solve $x^2+6x=7$, we add to each member such a number as will make the first member a perfect square. Plainly, 9 is such a number. Therefore $x^2+6x+9=16$. Next, take the square root of each member, and get $x+3=\pm 4$, whence $x=1$ or -7.

244. Similarly, to solve *any* affected quadratic equation, we first reduce the equation to a form where the terms containing x^2 and x are in the first member and the term not containing x is in the second member; second, if the coefficient of x^2 is not unity, divide each member of the equation by that coefficient, so that the coefficient of x^2 shall be unity; third, add to each member of the equation such a number as will make the first member a perfect square, and then take the square root of each member and solve the resulting simple equations.

245. Adding to a given expression a number that will make the sum a perfect square is called **Completing the Square.**

246. When the coefficient of x^2 is unity what number is it that we must add to each member of an equation to make the first member a perfect square? To answer this let us see how a perfect square is produced. We know that

$$(x+a)^2=x^2+2ax+a^2,$$

and $$(x-a)^2=x^2-2ax+a^2.$$

Notice here that whatever number is represented by a the third term is the square of one-half the coefficient of x; hence the number to be added to each member of the given equation is the square of one-half the coefficient of x.

247. Hence, to solve any affected quadratic equation:

I. Reduce to a form where both x^2 and x are in the first member and all terms not containing x are in the second member.

II. If the coefficient of x^2 is not already unity, divide each member of the equation by that coefficient, thus making the coefficient of x^2 unity.

III. Complete the square by adding to each member the square of one-half the coefficient of x.

IV. Extract the square root of each member of the equation and solve the resulting simple equations.

EXAMPLES.

Solve the following equations:

1. $x^2+4x=5$.
2. $x^2+6x=16$.
3. $2x^2+20x=48$.
4. $x^2+3x=18$.
5. $x^2+5x=36$.
6. $3x^2+6x=9$.
7. $4x^2-4x=8$.
8. $x^2-7x=-6$.
9. $x^2-10x=-9$.
10. $2x^2-15x=50$.
11. $x^2+8x=-15$.
12. $3x^2+12x=36$.
13. $2x^2+10x=100$.
14. $x^2-ax=6a^2$.
15. $x^2-2ax=3a^2$.
16. $3x^2-12ax=63a^2$.
17. $4x^2-12ax=16a^2$.
18. $5x^2-25x=-20$.
19. $x^2-x=2$.
20. $x^2+x=a^2+a$.

248. The method already given will enable us to solve any affected quadratic equation that may be given, but frequently it will oblige us to use fractions, and unless the terms of the fractions are small numbers it will be easier to complete the square by another method, which we will now consider.

249. We know that

$$(ax+b)^2=a^2x^2+2abx+b^2,$$

and

$$(ax-b)^2=a^2x^2-2abx+b^2,$$

so that each of these two second members is a perfect square. We therefore seek to reduce the given equation so that the first member shall be in the form of one of these two second members.

250. Notice two things: first, that the coefficient of x^2 is a perfect square, and second, that the third term equals the square of the quotient obtained by dividing the second term by twice the square root of the first term. Therefore, to reduce the first member of any given quadratic equation to one of the two forms

$$a^2x^2+2abx+b^2$$

or

$$a^2x^2-2abx+b^2,$$

I. Reduce the equation so that the terms containing x^2 and x shall be in the first member and all terms not containing x shall be in the second member.

II. Multiply each member of the equation by such a number as will make the coefficient of x^2 some perfect square.

III. Add to each member the square of the quotient obtained by dividing the second term by twice the square root of the first term.

The rest of the process of solution is like that already given, viz.: *take the square root of each member and solve the resulting simple equations.*

Let us solve by this method the equation

$$3x^2+7x+11=2x+33.$$

Transpose $2x$ and 11 and we get

$$3x^2+5x=22.$$

Multiply each member by 3 or 12 or 27 or 3 times any square number and the coefficient of x^2 will be a perfect square. Take the first of these multipliers and we get

$$9x^2+15x=66.$$

Add to each member $(\frac{15}{6})^2$, or $(\frac{5}{2})^2$, and we get

$$9x^2+15x+\tfrac{25}{4}=66+\tfrac{25}{4}=\tfrac{289}{4}.$$

Take the square root of each member and we get

$$3x+\tfrac{5}{2}=\pm\tfrac{17}{2}.$$

Hence, $3x=6$ or -11, and $x=2$ or $-\frac{11}{3}$.

If we had multiplied by 12 instead of 3 we would have obtained $36x^2+60x=264$.

Therefore, $36x^2+60x+25=264+25=289$.

Hence, $6x+5=\pm17$.

Hence, $6x=12$ or -22, and $x=2$ or $-\frac{11}{3}$.

EXAMPLES.

Solve the following equations:

21. $3x^2+4x=7$.
22. $3x^2+6x=24$.
23. $4x^2-5x=26$.
24. $9x^2+6x-48=0$.
25. $18x^2-3x-66=0$.
26. $5x^2-7x=24$.
27. $2x^2-35=3x$.
28. $3x^2-50=5x$.
29. $\frac{1}{2}x^2-3x+\frac{17}{18}=0$.
30. $\dfrac{x^2}{4}-\dfrac{3x}{2}+2=0$.
31. $3x^2-24=6x$.
32. $2x^2-22x=-60$.
33. $2x^2+10x=300$.
34. $3x^2-10x=200$.
35. $2x^2-3x=104$.
36. $3x^2+7x-370=0$.
37. $4x^2-7x+\frac{5}{2}=0$.
38. $5x^2-\frac{1}{2}x-\frac{7}{18}=0$.
39. $\frac{2}{3}x^2-\frac{3}{2}x=-\frac{19}{7}$.
40. $\dfrac{x^2}{3}-\dfrac{x}{2}+\dfrac{1}{6}=0$.

251. Literal quadratic equations may be solved the same as numerical ones. But, after the square is completed, the second member will be a literal instead of a numerical expression, and hence the square root usually cannot be taken, and so will have to be *indicated*. Thus, to solve $x^2+4ax+b=0$, we proceed as follows:

Transpose b, $\quad x^2+4ax=-b.$

Add $4a^2$ to each member,

$$x^2+4ax+4a^2=4a^2-b.$$

Take the square root of each member,

$$x+2a=\pm\sqrt{4a^2-b}.$$

Transpose $2a$. $\quad x=-2a\pm\sqrt{4a^2-b}.$

One value of x is $-2a+\sqrt{4a^2-b}$ and the other value is $-2a-\sqrt{4a^2-b}$.

EXAMPLES.

Solve the following equations:

41. $x^2+2ax=b.$
42. $x^2+4ax=b.$
43. $2x^2+3ax=4b.$
44. $x^2-5ax=2b.$
45. $x^2-6ax-3b=0.$
46. $2x^2-6ax-4b=0.$
47. $5x^2-7ax+b=0.$
48. $ax^2+bx+c=0.$
49. $ax^2-bx=c.$
50. $ax^2+a^2x=a^3.$

252. When all the terms of an affected quadratic equation are transposed to the left member, making the right member zero, the equation may be solved by factoring if the resulting first member can readily be expressed as the product of two factors each of the first degree with respect to the unknown numbers. Thus, to solve the equation $x^2-5x=-6$, we transpose -6, and obtain $x^2-5x+6=0$. By the method of exercise 88 we find the factors of the left member to be $x-2$ and $x-3$. Therefore the equation may be written $(x-2)(x-3)=0$, which is satisfied if $x=2$ or if $x=3$.

253. This method of solving an affected quadratic cannot be used to advantage unless the left member is easily factored, but when easily factored this is probably the easiest way to solve them.

EXAMPLES.

Solve by factoring the following equations.

51. $x^2+5x+6=0$.

52. $x^2+11x=-30$.

53. $x^2-x=6$.

54. $x^2+7x=-12$.

55. $x^2-11x+30=0$.

56. $x^2-5x=14$.

57. $x^2+x=30$.

58. $x^2+x=12$.

59. $x^2-7x+12=0$.

60. $2x^2+6x=x^2-3x-14$.

61. $2x^2+3x+4=x^2-3x-1$.

62. $x^2+2x+1=6x+6$.

63. $3x^2+12x-10=2x^2+2x-31$.

64. $x^2=6x-5$.

65. $x^2=-4x+21$.

66. $x^2=-4x+5$.

67. $x^2-49=10(x-7)$.

68. $2x^2+60x=-400$.

69. $10x^2+600x=-8000$.

70. $2x^2-40x=4x-240$.

71. $x^2-(a+b)x+ab=0$.

72. $x^2-(a+1)x+a=0$.

73. $x^2+(a+b)x+ab=0$.

74. $x^2+(a+1)x+a=0$.

75. $x^2+(ab+1)x+ab=0$.

EXERCISE 103.

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

254. To solve a problem the first thing to do is to form an equation the result of solving which will fulfill all the requirements of the problem. The formation of this equation is sometimes a great difficulty to students, but after the equation is once written down there is

usually little or no difficulty with a problem. The difficulty here spoken of may be largely overcome if the student will keep in mind the fact that the equation is formed by using the unknown number in exactly the same way we would use any assumed result to see whether or not this assumed result were right. This is illustrated in the following problem.

1. A train travels 600 miles at a uniform rate of speed. If the rate had been 10 miles more an hour the journey would have taken 5 hours less. Find the rate of the train.

Let us see if 40 miles an hour is the rate. If the train goes 40 miles an hour, to go 600 miles will require $\frac{600}{40}$ or 15 hours. And if the rate were 10 miles more an hour it would be 50 miles an hour, and to travel 600 miles at this rate would require $\frac{600}{50}$ or 12 hours. By the first supposition (40 miles an hour) it requires 15 hours, and by the second supposition (50 miles an hour) it requires 12 hours. As this last result (12 hours) is not 5 hours less than the first result (15 hours) we conclude that the rate is *not* 40 miles an hour.

Now let us see if 30 miles an hour is the rate. To travel 600 miles at 30 miles an hour requires $\frac{600}{30}$ or 20 hours, and to travel 600 miles at 40 miles an hour requires $\frac{600}{40}$ or 15 hours; and as $20-15=5$, (*i. e.* the second time is 5 hours less than the first,) we conclude that the rate *is* 30 miles an hour.

Now to form an equation we say, let x represent the number of miles an hour the train travels; then to travel 600 miles at x miles an hour requires $\frac{600}{x}$ hours, and to travel 600 miles at $x+10$ miles an hour requires $\frac{600}{x+10}$ hours, and as the time in the second instance is 5 hours less than in the first instance, we have

$$\frac{600}{x}-\frac{600}{x+10}=5.$$

From this equation we have

$$5x^2+50x=600(x+10)-600x,$$

or $$5x^2+50x=6000,$$

or $$x^2+10x=1200.$$

Completing the square, we have

$$x^2+10x+25=1225.$$

Extracting the square root, we have

$$x+5=\pm 25.$$

Hence $$x=30 \text{ or } -40.$$

The first of these results, 30, agrees with the conclusion reached above, but here another question arises,—what is to be done with the result -40? So far as the algebraic work goes, -40 is as good a result as 30, but to speak of a train traveling -40 miles an hour is something void of meaning, so this result is rejected and 30 is retained as the true result.

255. It will often happen, as in the example just worked, that the solution of the equation formed as already described leads to a result which does not apply to the problem we are solving. The reason of this is that the algebraic statement of the problem (by means of the equation formed as above described) is more general than the statement in words. It will, however, usually be quite easy to select which result belongs to the problem we are solving, and then we can reject the other result as inapplicable.

2. A cistern can be filled by two pipes in $33\frac{1}{3}$ minutes. If the smaller pipe takes 15 minutes more than the larger one to fill the cistern, in what time will it be filled by each pipe singly?

Let us see if the smaller pipe would fill the cistern in 45 minutes. If so, the larger pipe would fill the cistern in 30 minutes. If larger pipe will fill the cistern in 30 minutes it would fill $\frac{1}{30}$ of the cistern in 1 minute, and if smaller pipe will fill the cistern in 45 minutes it would fill $\frac{1}{45}$ of the cistern in 1 minute. Therefore, together they would fill $\frac{1}{30}+\frac{1}{45}=\frac{1}{18}$ of cistern in 1 minute. But together they fill $\frac{1}{33\frac{1}{3}}=\frac{3}{100}$ of cistern in 1 minute, and as $\frac{1}{18}$ is not equal to $\frac{3}{100}$ we conclude that 45 minutes is *not* the time the smaller pipe would fill the cistern.

Let x=number of minutes required to fill cistern by smaller pipe, then $x-15$=number of minutes required to fill cistern by larger pipe.

If larger pipe will fill the cistern in $x-15$ minutes it would fill $\frac{1}{x-15}$ of cistern in 1 minute, and if smaller pipe would fill cistern in x minutes it would fill $\frac{1}{x}$ of cistern in 1 minute. Therefore the two pipes together would fill $\frac{1}{x-15}+\frac{1}{x}$ or $\frac{2x-15}{x(x-15)}$ of cistern in 1 minute. But together they fill $\frac{1}{33\frac{1}{3}}$ or $\frac{3}{100}$ of cistern in 1 minute. Therefore

$$\frac{3}{100}=\frac{2x-15}{x(x-15)}.$$

Clearing of fractions, we get

$$3x^2-45x=200x-1500.$$

Transposing $200x$, $\quad 3x^2-245x=-1500.$

Multiply by 12, $\quad 36x^2-2940x=-18000.$

Add $(245)^2$ to each member to complete the square,

$$36x^2-2940x+60025=42025.$$

Extract the square root of each member,

$$6x-245=\pm 205.$$

Hence, $\quad 6x=450$ or $40.$

Therefore, $\quad x=75$ or $6\frac{2}{3}.$

From this it appears that the smaller pipe would fill the cistern in either 75 or $6\frac{2}{3}$ minutes, but evidently $6\frac{2}{3}$ is not admissible, for it takes the smaller pipe 15 minutes *more* to fill the cistern than it takes the larger pipe *some* time to fill the cistern. So it is plain that it must take the smaller pipe *more than* 15 *minutes* to fill the cistern. We therefore reject the result $6\frac{2}{3}$ as being inadmissible; the other result, 75, is admissible, however, and satisfies the requirements of the problem. Hence, it takes the smaller pipe 75 minutes, and therefore the larger one 60 minutes to fill the cistern alone.

3. A merchant sold some damaged goods for \$72, which was a loss per cent. of $\frac{1}{3}$ of the number of dollars the goods cost. Find the cost of the goods.

Let $\quad x$=number of dollars goods cost,

then $\quad \frac{x}{3}$=loss per cent.

Therefore, $\quad \frac{x^2}{300}$=entire loss

By the statement in the problem we have

$$\frac{x^2}{300}+72=x.$$

$$x^2+21600=300x,$$

$$x^2-300x=-21600,$$

$$x^2-300x+22500=900,$$

$$x-150=\pm 30,$$

$$x=180 \text{ or } 120.$$

Each of these answers fulfills all the requirements of the problem, and each is admissible.

4. One of two numbers is $\frac{2}{3}$ the other one and the sum of their squares is 208. Find the two numbers.

5. The product of two numbers is 750 and the quotient of the greater divided by the less is $3\frac{1}{3}$. Find the numbers.

6. Divide the number 100 into two parts whose product is 2400.

7. Divide the number 60 into two such parts that the quotient of the greater divided by the less may equal 1 more than twice the less.

8. A merchant bought a quantity of cloth for \$120; if he had bought 6 yards more for the same sum the price per yard would have been \$1 less. How many yards did he buy, and what was the price per yard?

9. A merchant sold some goods for \$39, and in so doing gained as much per cent. as the goods cost him. What was the cost of the goods?

10. Find a number such that 3 more than twice the number multiplied by 3 less than twice the number may give a product of 112.

11. A man traveled 105 miles, and then found if he had gone 2 miles less per hour he would have been 6 hours longer on his journey. How many miles did he travel per hour?

12. A man bought two farms for $2800 each; the larger contained 10 acres more than the smaller, but he paid $5 more per acre for the smaller than for the larger. How many acres were there in each farm?

13. A merchant sold two pieces of cloth which together contained 40 yards, and received for each piece twice as many cents per yard as there were yards in the piece. For the smaller piece he received $\frac{9}{25}$ as much as for the larger one. How many yards were there in each piece?

14. The distance around a rectangle is 200 feet, and the area is 1344 square feet. Find the length and breadth of the rectangle.

15. The length of a rectangle is 10 feet more than the breadth, and the area is 600 square feet. Find the length and breadth of the rectangle.

16. There are three lines, the first two of which are $\frac{4}{7}$ of the third, and the sum of the squares described on these three lines is 33 square feet. Find the lengths of these lines.

17. A flower bed 9 feet long and 6 feet wide has a path around it whose area is equal to the area of the bed itself. What is the width of the path?

18. A number consists of two digits one of which is the square of the other, and if 54 be added to the number the digits are reversed in order. What is the number?

19. Find a number such that if it be added to 94 and again subtracted from 94 the product of the sum and difference thus obtained shall be 8512.

20. A man bought a number of horses for $10000; each cost 4 times as many dollars as there were horses. How many horses did he buy?

21. Find a number whose square is greater than the number itself by 306.

22. Find a number such that if its third part be multiplied by its fourth part and to the product 5 times the number be added the sum exceeds 200 by as much as the number required is less than 280.

23. A man bought a horse and sold it again for $119, by which means he gained as many per cent. as the horse cost him dollars. How many dollars did the horse cost him?

24. Find the fortunes of three persons, A, B, and C, from the following data: For every $5 which A has B has $9 and C has $10; moreover, if we multiply A's money by B's, and B's money by C's, and add both products to the united fortunes of all three we shall get $8832. How much money has each?

25. The combined area of two squares is 962 square feet, and a side of one square is 18 feet longer than a side of the other. What is the size of each square?

26. A square field contains a number of square rods equal to 260 more than 32 times its perimeter. How many rods in one side of the square?

27. Find two numbers whose sum is 21 and the sum of whose squares is 225.

28. Find two numbers whose product is 480 and the difference of whose squares is 3536.

29. What is the price of oranges when 10 more for $1.20 lowers the price 1 cent each?

30. The sum of the ages of a father and son is 85 years, and $\frac{1}{3}$ of the product of their ages in years exceeds 5 times the father's age by 200 years. What is the age of each?

31. A certain number is the product of three consecutive whole numbers, and if it is divided by each one of these three factors in turn the sum of the three quotients thus obtained is 767. What is the number?

32. The sum of the squares of three consecutive odd numbers is 83. What are the numbers?

33. The sum of the squares of four consecutive even numbers is 120. What are the numbers?

34. Divide the number 18 into two such parts that their product shall exceed 30 times their difference by 20.

35. In a bag which contains coins of silver and gold each silver coin is worth as many cents as there are gold coins, and each gold coin is worth as many dollars as there are silver coins, and the whole is worth $525. How many gold and how many silver coins in the bag?

36. A room whose length exceeds its breadth by 8 feet is covered with matting 4 feet wide, and the number of yards in length of the matting exceeds $\frac{4}{3}$ the number of feet in breadth of the room by 20. Find the length and breadth of the room.

37. There are two numbers whose difference is 7, and half their product, plus 30, is equal to the square of the smaller number. What are the numbers?

38. A and B start together on a journey of 36 miles. A travels 1 mile per hour faster than B and arrives 3 hours before him. Find the rate of each.

39. Two workmen, A and B, are engaged to work at different wages. A works a certain number of days and receives $27, and B, who works 1 day less than A, receives $34. If A had worked 2 days more and B 2 days less, they would have received equal amounts. Find the number of days each worked.

EXERCISE 109.

EQUATIONS SOLVED LIKE QUADRATICS.

256. It is beyond the scope of this book to treat equations of a higher degree than the second expressed in general form, but there are a few equations of higher degree which may be solved by the methods of this chapter.

257. We have had such equations as $x^2-13x+36=0$, and have seen that such equations are easily solved. Now it is plain that we could use some other symbol in place of x to designate an unknown number. Thus we might have an equation where y^2 stands in place of x, and then of course y^4 would stand in place of x^2, and the equation would be

$$y^4-13y^2+36=0,$$

from which, by solving in the usual way, regarding y^2 temporarily as the unknown number, we obtain

$$y^2=4 \text{ or } 9.$$

Hence $$y=\pm 2 \text{ or } \pm 3.$$

In a similar manner we could treat equations in which more complex expressions stand in place of x^2 and x in the equations before used, but whatever expression stands in place of x the square of that expression must stand in place of x^2, else the equation cannot be solved by the methods of this chapter.

EXAMPLES.

Solve the following equations:

1. $x^4-29x^2+100=0$.
2. $x^6-35x^3+216=0$.
3. $y^4-17y^2+16=0$.
4. $y+81\sqrt{y}+15=0$.
5. $(x+3)^6-28(x+3)^3+27=0$.

6. $x^2+\frac{1}{x^2}=a^2+\frac{1}{a^2}$.

7. $(x^2-x)^2-\frac{27}{4}(x^2-x)+\frac{9}{2}=0$.

8. $x^2+\frac{100}{x^2}=29$.

9. $\left(x+\frac{1}{x}\right)^2-\frac{10}{3}\left(x+\frac{1}{x}\right)+1=0$.

10. $(x^2-5x+6)^2=14(x^2-5x+6)-24=0$.

EXERCISE 110.

THEORY OF QUADRATIC EQUATIONS.

253. The methods already explained will enable us to solve a quadratic equation in which letters are used to stand for the known numbers in the equation, but of course we must not use the same letter to stand for a known number that we use to stand for an unknown number.

Let us solve the equation $x^2+ax+b=0$.

Transpose b, $\quad x^2+ax=-b$.

Complete square,

$$x^2+ax+\frac{a^2}{4}=\frac{a^2}{4}-b=\frac{a^2-4b}{4}.$$

Extract square root, $x+\frac{a}{2}=\pm\sqrt{\frac{a^2-4b}{4}}$.

Hence, $\quad x=-\frac{a}{2}\pm\sqrt{\frac{a^2-4b}{4}}$.

From this we see that one root of the given equation

$$=-\frac{a}{2}+\sqrt{\frac{a^2-4b}{4}},$$

and the other root of the given equation

$$=-\frac{a}{2}-\sqrt{\frac{a^2-4b}{4}}.$$

259. It thus appears that there are *two* roots to a quadratic equation, but for certain values of a and b these two roots are exactly the same. This will be the case when $a^2-4b=0$, for then the number under the radical sign reduces to zero, and each root of the equation reduces to $-\frac{a}{2}$. In this case there is in reality only *one* value of x that will satisfy the equation. Still, instead of saying that there is only one root of the equation, we say that there are *two* roots but that these two are *equal to each other*. Of course this is only another way of saying that there is only one root, but the advantage of this mode of expression will be apparent as we proceed.

260. Let us now find the sum of the roots of the equation $x^2+ax+b=0$.

$$\text{One root}=-\frac{a}{2}+\sqrt{\frac{a^2-4b}{4}}$$

$$\text{other root}=-\frac{a}{2}-\sqrt{\frac{a^2-4b}{4}}$$

$$\text{sum}=-a$$

In the quadratic equation $x^2+ax+b=0$ the coefficient of x^2 is unity, but a and b stand for *any* numbers. Hence in *any* quadratic equation where all the significant terms are in the left member and the right member is 0, and where the coefficient of x^2 is unity, *the sum of the roots is equal to the coefficient of x with its sign changed.*

261. Let us now find the product of the roots of the equation $x^2+ax+b=0$.

The product required may be expressed thus:

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2-4b}{4}}\right)\left(-\frac{a}{2}-\sqrt{\frac{a^2-4b}{4}}\right)$$

Notice that this is the product of the sum and difference of two numbers, which, as we have already learned, is equal to the difference of their squares. Therefore the product of the two roots equals

$$\frac{a^2}{4}-\left(\frac{a^2-4b}{4}\right)=b.$$

Hence in any quadratic equation where all the significant terms are in the left member and the right member is 0, and where the coefficient of x^2 is unity, *the product of the roots is equal to the term not containing* x.

262. The last two articles show just the relation between the known numbers in an equation and the roots of the equation, and from the results reached we are enabled to form an equation which shall have any desired roots. For example, if we wish to form the equation whose roots are 3 and 5, we know that the coefficient of x^2 will be 1, the coefficient of x with its sign changed will be the sum of the roots, *i. e.*, in this case the coefficient of x will be -8, and the term not containing x will be the product of the roots, *i. e.*, in this case the term not containing x will be 15. Hence the equation is

$$x^2-8x+15=0.$$

EXAMPLES.

Form the equations whose roots are the following given numbers:

1. 2 and 3.
2. 3 and 6.
3. 1 and 7.
4. 6 and -6.
5. 3 and 3.
6. 4 and 6.
7. 3 and -1.
8. 2 and 2.
9. -4 and -4.
10. 10 and 20.
11. 5 and -3.
12. -7 and -5.
13. 12 and -12.
14. $\frac{2}{3}$ and $\frac{3}{2}$.
15. 2 and 0.

263. In Art. 258 we solved the equation $x^2+ax+b=0$ and obtained for the value of x

$$-\frac{a}{2}\pm\sqrt{\frac{a^2-4b}{4}}.$$

This result may be used as a *formula* for finding the roots of any quadratic equation of the form $x^2+ax+b=0$, *i. e.*, any quadratic equation in which the coefficient of x^2 is unity and all the terms are in the left member. To obtain the roots of any quadratic equation of this form we have only to substitute in the place of a and b in the result given above their numerical values. Thus we may obtain the roots of $x^2+3x-70=0$ by writing 3 in place of a and -70 in place of b. Making this substitution, we get for the roots

$$-\frac{3}{2}\pm\sqrt{\frac{9+280}{4}},$$

$$=-\tfrac{3}{2}\pm\tfrac{17}{2}=7 \text{ or } -10.$$

EXAMPLES.

In this manner find the roots of the following equations:

16. $x^2+12x+11=0$.
17. $x^2-12x+5\frac{3}{4}=0$.
18. $x^2+15x+31\frac{1}{4}=0$.
19. $x^2+9x-43\frac{3}{4}=0$.
20. $x^2-20x-44=0$.
21. $x^2+16x+28=0$.
22. $x^2+3x-1\frac{3}{4}=0$.
23. $x^2-2x+\frac{3}{4}=0$.
24. $x^2+7x+6=0$.
25. $x^2-5x-2\frac{3}{4}=0$.

264. Let us try to use the same method to find the roots of the equation $x^2+2x+8=0$.

By substituting, we have for the roots

$$-1\pm\sqrt{\frac{4-32}{4}}=-1\pm\sqrt{-7}.$$

But there is no number which squared will give -7, and so -7 has no square root. Nevertheless, such expressions as $\sqrt{-7}$ do frequently occur in Algebra, and are called *impossible numbers* or *imaginary expressions.*

265. An **Imaginary Expression** is any expression which contains one or more terms in which there is an indicated even root of a negative number. By *even* root we mean the *square* root, *fourth* root, *sixth* root, etc. Thus

$$\sqrt{-5}, \quad 4+\sqrt{-2}, \quad a+b-\sqrt{-5},$$

are imaginary expressions.

266. By way of distinction those expressions which are not imaginary are called **Real.** All of the expressions with which we have had to deal heretofore have been real.

267. By inspecting the expressions for the roots of the equation $x^2+ax+b=0$ we can tell when the two roots are real and when imaginary, for the roots will be real when the expression under the radical sign is positive, and imaginary when that expression is negative. The expression under the radical is $\frac{a^2-4b}{4}$ and in this expression a^2 must always be *positive* because it is the *square* of some number; so it is plain that the expression $\frac{a^2-4b}{4}$ is positive when a^2 is greater than $4b$, and negative when a^2 is less than $4b$. Hence, the roots are real when a^2 is greater than $4b$, and imaginary when a^2 is less than $4b$.

If the quadratic equation that is given is in a different form, we can still find when the roots are real and when imaginary just as easily as was done in the case just considered.

Take, for example, the equation $ax^2+bx+c=0$.

Solving this, we get

$$x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{b^2-4ac};$$

and here the expression under the radical sign is positive when b^2 is greater than $4ac$, and negative when b^2 is less than $4ac$. Therefore, the roots are real when b^2 is greater than $4ac$, and imaginary when b^2 is less than $4ac$.

EXAMPLES.

Solve the following equations and determine when the roots are real and when imaginary:

26. $x^2-2ax+2b=0$.

27. $ax^2-2bx+3c=0$.

28. $x^2-2\frac{b}{a}x-\frac{3c}{a}=0$.

29. $2x^2+3ax=5b$.

30. $ax^2+6bx+9ab=0$.

31. $ax^2-4bx-4=0$.

32. $x^2-4ax+5=0$.

33. $x^2-4x+a=0$.

34. $a^2x^2+4bx-4c=0$.

35. $2ax^2+3bx-4abc=0$.

268. By inspecting the roots of equations we may determine other things, than when the roots are real and when imaginary; for example, we may determine when the roots are equal to each other.

The roots of the equation $x^2+ax+b=0$ have already been found to be

$$-\frac{a}{2}\pm\sqrt{\frac{a^2-4b}{4}}.$$

Whatever be the value of $\sqrt{\frac{a^2-4b}{4}}$ we must *add* its value to $-\frac{a}{2}$ to obtain *one* root of the equation and *subtract* it from $-\frac{a}{2}$ to obtain the *other* root, and plainly the *only*

way these two results *can* be just the same, is for the radical part to be 0, *i. e.* for the expression under the radical sign to be 0. Therefore, in order that the roots of

$$x^2+ax+b=0$$

may be equal to each other, we must have

$$a^2-4b=0, \text{ or } a^2=4b.$$

If the given quadratic equation had been in a different form we could solve it, and by inspecting the result, tell when the two roots were equal to each other; for plainly, *in any case* for the roots to be equal, the term preceeded by the double sign $\pm$ must be 0.

36. Determine when the roots of each of the equations in examples 26 to 35 are equal to each other.

269. Still other questions about the roots might be answered,—as when are both roots *positive*, when *negative*, when is *one positive* and the *other negative*, etc.,—but the cases given are enough to show the student that much may be learned by inspecting the result obtained through solving a literal equation.

CHAPTER XV.

SIMULTANEOUS EQUATIONS ABOVE THE FIRST DEGREE.

EXERCISE 111.

ONE EQUATION OF THE FIRST AND ONE OF THE SECOND DEGREE.

270. In the solution of simultaneous equations above the first degree we have before us the same general problem as in the solution of simultaneous equations of the first degree, viz.: to find by some combinations of the equations given those values of the unknown numbers which are common to all the given equations, or what is the same thing, those values of the unknown numbers which satisfy all the given equations at the same time.

271. In this chapter we confine our attention to the case of *two* simultaneous equations with two unknown numbers.

The *general* case of two simultaneous equations above the first degree cannot be solved without knowing how to solve a single equation of a higher degree than the second, which we are not now supposed to know how to do, so the *general* case cannot be taken up; but there are two cases where the solution is quite easy. Other cases than these two depend more or less upon the ingenuity of the student, and often require some special device for their solution.

272. We take as our *First Case* that in which one equation is of the first degree and the other of the second degree.

Let us find the values of x and y from the equations

$$x^2+y=11 \quad (1)$$
$$x+y=5 \quad (2)$$

From (2), $y=5-x.$

Substitute this value in (1) and we get

$$x^2+5-x=11.$$

Therefore, $x^2-x=6,$

$$x^2-x+\tfrac{1}{4}=\tfrac{25}{4},$$
$$x-\tfrac{1}{2}=\pm\tfrac{5}{2},$$
$$x=3 \text{ or } -2.$$

Substituting the first of these values in (2),

$$3+y=5.$$

From which, $y=2.$

Substituting the second value of x in (2),

$$-2+y=5.$$

From which, $y=7.$

Therefore, $\begin{cases} x=3 \text{ or } -2. \\ y=2 \text{ or } 7. \end{cases}$

These values go together in pairs, *i. e.*, the value 3 of x goes with the value 2 of y, and the value -2 of x goes with the value 7 of y.

273. If we now look to see just what was done in the solution just given, we will observe that from the equation of the *first* degree the value of one unknown number was found in terms of the other unknown number, and this value was substituted in the equation of the *second* degree. This method is capable of solving *any* case where one equation is of the first and one of the second degree.

Hence, to find the values of two unknown numbers from two equations, one of which is of the first and the other of the second degree, we proceed as follows:

From the equation of the first degree find the value of either unknown number in terms of the other unknown number; substitute the value so found in place of this unknown number in the equation of the second degree, and solve the resulting quadratic equation. This gives two values for one of the unknown numbers, which values substitute in turn in the equation of the first degree, and thus find the values of the other unknown number.

EXAMPLES.

Solve the following sets of equations:

1. $\begin{cases} x^2+3y^2=52. \\ 2x-3y=1. \end{cases}$

2. $\begin{cases} x^2+xy+y^2=61. \\ x-y=1. \end{cases}$

3. $\begin{cases} 3xy-4y^2=20. \\ 3x-4y=10. \end{cases}$

4. $\begin{cases} x+y=6. \\ xy=5. \end{cases}$

5. $\begin{cases} x^2+3xy-y^2=3. \\ 3x-y=1. \end{cases}$

6. $\begin{cases} x^2+y^2+2x=31. \\ x-y+1=0. \end{cases}$

7. $\begin{cases} xy+x+y=32. \\ 4x+8y=56. \end{cases}$

8. $\begin{cases} 3x^2-2xy+y^2=81. \\ 10x-2y=54. \end{cases}$

9. $\begin{cases} 2y^2+xy=14. \\ 2y+x=7. \end{cases}$

10. $\begin{cases} x^2-y^2=-9. \\ x-y=-1. \end{cases}$

11. $\begin{cases} x^2+y^2=2. \\ x+y=2. \end{cases}$

12. $\begin{cases} 5x^2-3xy-2y^2=12. \\ 2x-y=3. \end{cases}$

13. $\begin{cases} 2x^2-3xy-4y^2=16. \\ 2x-4y=4. \end{cases}$

14. $\begin{cases} x^2-xy=320. \\ 3x+12y=12. \end{cases}$

15. $\begin{cases} xy+x+y=19. \\ 2x+3y=18. \end{cases}$

16. $\begin{cases} x^2+y^2+x+y=152. \\ 2x-y=2. \end{cases}$

17. $\begin{cases} 6x^2-y^2+8x-2y=-225. \\ 4x-y=5. \end{cases}$

18. $\begin{cases} 4x^2-4y^2+x-y=82. \\ 4x+4y=40. \end{cases}$

19. $\begin{cases} 4x^2+5xy-6y^2-7x-8y=16. \\ x-2y=1. \end{cases}$

20. $\begin{cases} 10x^2-9y^2+5x+4y=-569. \\ 5x-5y=-15. \end{cases}$

21. $\begin{cases} x^2+2xy+3y^2+x=36. \\ 2x+3y=12. \end{cases}$

22. $\begin{cases} \frac{2}{x+2}+\frac{1}{y+1}=2. \\ y-x=1. \end{cases}$

23. $\begin{cases} \frac{1}{x}-y=2. \\ x+5y=5\frac{1}{3}. \end{cases}$

24. $\begin{cases} \frac{1}{x}-\frac{1}{y}=-3\frac{1}{2}. \\ x-y=1\frac{3}{4}. \end{cases}$

25. $\begin{cases} \frac{5}{x}+\frac{5}{y}+\frac{y}{x}+\frac{x}{y}=4. \\ 5x+5y=50. \end{cases}$

26. $\begin{cases} \frac{x}{y}-\frac{y}{x}=0. \\ x-2y=-6. \end{cases}$

27. $\begin{cases} x+y+\frac{5}{x+y}=6. \\ 3x-y=3. \end{cases}$

28. $\begin{cases} x+y+\frac{2}{x-y}=7. \\ x-y=1. \end{cases}$

29. $\begin{cases} x+y+\frac{14}{x+y}=9. \\ 2x-5y=-7. \end{cases}$

30. $\begin{cases} \frac{x}{x+y}+\frac{y}{x-y}-\frac{2x+y}{x^2-y^2}=-3. \\ x-y=1. \end{cases}$

EXERCISE 112.

TWO QUADRATIC EQUATIONS.

274. The *Second Case* is that in which each equation is of the second degree and possesses this peculiarity, viz : that all the terms which contain unknown numbers are of the second degree with respect to those numbers.

Let us solve the equations

$$x^2+y^2=25 \quad (1)$$

$$x^2+xy-y^2=19 \quad (2)$$

Substitute vx in place of y, and we obtain

$$x^2+v^2x^2=25 \quad (3)$$

$$x^2+vx^2-v^2x^2=19 \quad (4)$$

From (3) we obtain $$x^2=\frac{25}{1+v^2}. \quad (5)$$

From (4) we obtain $$x^2=\frac{19}{1+v-v^2}. \quad (6)$$

Therefore, $$\frac{25}{1+v^2}=\frac{19}{1+v-v^2}.$$

Clearing of fractions, we get

$$25(1+v-v^2)=19(1+v^2),$$

$$25+25v-25v^2=19+19v^2,$$

$$44v^2-25v=6,$$

$$v^2-\tfrac{25}{44}v=\tfrac{6}{44},$$

$$v^2-\tfrac{25}{44}v+\tfrac{625}{7744}=\tfrac{6}{44}+\tfrac{625}{7744}=\tfrac{681}{7744},$$

$$v-\tfrac{25}{88}=\pm\tfrac{81}{88},$$

$$v=\tfrac{66}{88} \text{ or } -\tfrac{16}{88},$$

$$v=\tfrac{3}{4} \text{ or } -\tfrac{2}{11}.$$

Substituting each of these values in turn for v in (5),

$$x=\pm 4 \text{ or } \pm\frac{11}{\sqrt{5}}.$$

And since $y=vx$, we get

$$y=\pm 3 \text{ or } \mp\frac{2}{\sqrt{5}}.$$

These values go together in pairs as indicated in the following scheme, where any value given for x goes with that value of y which is *directly underneath* the value of x considered:

$$x=4 \text{ or } -4 \text{ or } \frac{11}{\sqrt{5}} \text{ or } -\frac{11}{\sqrt{5}}.$$

$$y=3 \text{ or } -3 \text{ or } -\frac{2}{\sqrt{5}} \text{ or } \frac{2}{\sqrt{5}}.$$

275. If we look carefully at the solution just given we will observe that vx was substituted for y in each of the given equations, and from each of the resulting equations the value of x^2 was found in terms of v. These two values of x^2 were placed equal to each other, and the resulting equation solved to find the value of v. This value of v was then substituted in the first of the equations which expressed the value of x^2 in terms of v, and thence the value of x^2 and then the value of x was determined. The value found for x together with the value found for v served to determine y.

276. The method used in solving the set of equations just given is capable of solving *any* two equations possessing the peculiarity mentioned in the last article. Hence the method may be stated as follows:

Substitute vx in place of y in both equations, and from each of the resulting equations find the value of x^2 in terms of v. Place these two values of x^2 equal to each other and solve the resulting quadratic equation to determine v. Substitute the value found for v in one of the first two equations where v was used, and thus determine the value of x. Having both x and v, determine y, which is the product of x and v.

EXAMPLES.

Solve the following sets of equations:

1. $\begin{cases} x^2+y^2=25. \\ xy=12. \end{cases}$

2. $\begin{cases} x^2-3xy+y^2=5. \\ x^2-2xy=8. \end{cases}$

3. $\begin{cases} 2x^2-3xy=0. \\ 3xy-2y^2=10. \end{cases}$

4. $\begin{cases} 2x^2-3y^2+xy=3. \\ 3xy+x^2=-2. \end{cases}$

5. $\begin{cases} 4x^2-5xy+y^2=7. \\ x^2-5y^2=-1. \end{cases}$

6. $\begin{cases} x^2+2xy+y^2=36. \\ x^2-2xy+y^2=0. \end{cases}$

7. $\begin{cases} 2x^2+3y^2=62. \\ 2x^2-3y^2=38. \end{cases}$

8. $\begin{cases} x^2+xy+y^2=61. \\ 3xy=60. \end{cases}$

9. $\begin{cases} x^2-4y^2=0. \\ xy-y^2=9. \end{cases}$

10. $\begin{cases} x^2-y^2=27. \\ 5xy=90. \end{cases}$

11. $\begin{cases} \frac{x}{y}+\frac{y}{x}=2\frac{1}{6}. \\ x^2+y^2=13. \end{cases}$

12. $\begin{cases} 2x^2-3xy+4y^2=48. \\ x+5y=\frac{192}{x+y}. \end{cases}$

13. $\begin{cases} \frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{10}{3}. \\ x^2+y^2=45. \end{cases}$

14. $\begin{cases} 4xy=40. \\ x^2-3y^2=13. \end{cases}$

15. $\begin{cases} x^2+3xy-2y^2=26. \\ x^2-3xy-y^2=3. \end{cases}$

16. $\begin{cases} 3x^2-xy=133. \\ xy-3y^2=2. \end{cases}$

17. $\begin{cases} x^2+y^2=25. \\ xy-y^2=-4. \end{cases}$

18. $\begin{cases} x^2-3xy-y^2=-29. \\ x^2-xy+y^2=13. \end{cases}$

19. $\begin{cases} \frac{3}{x^2}-\frac{4}{y^2}=\frac{8}{x^2y^2}. \\ \frac{2}{x^2}-\frac{3}{y^2}=\frac{2\frac{1}{2}}{xy}. \end{cases}$

20. $\begin{cases} \frac{x^2}{y}+3y-2x=\frac{36}{y}. \\ \frac{y^2}{x}-4x+3y=\frac{24}{x}. \end{cases}$

EXERCISE 113.

MISCELLANEOUS EQUATIONS.

277. Since no general directions can be laid down to cover all solvable cases of simultaneous quadratic equations, a few examples will be worked out as samples. The student is advised to study carefully the following solutions, and try to see, if possible, what suggested the various steps taken in the solution.

1. $\begin{cases} x^2+y^2=20. & (1) \\ x+y=6. & (2) \end{cases}$

Square (2), $x^2+2xy+y^2=36$ (3)

Put (1) $x^2+y^2=20$

Subtract, $2xy=16$ (4)

Subtract (4) from (1), $x^2-2xy+y^2=4$

Extract square root, $x-y=\pm 2$

But (2) $x+y=6$

Therefore, adding, $2x=8$ or 4

Hence, $x=4$ or 2

Substitute in (2), $y=2$ or 4

2. $\begin{cases} x+y=6. & (1) \\ xy=5. & (2) \end{cases}$

Square (1), $x^2+2xy+y^2=36$

Multiply (2) by 4, $4xy=20$

Subtract, $x^2-2xy+y^2=16$

Extract square root, $x-y=\pm 4$

But (2) $x+y=6$

Therefore, adding, $2x=10$ or 2

Hence, $x=5$ or 1

Substitute in (1), $y=1$ or 5

ANOTHER SOLUTION.

Let $y=vx$ in both equations; then $x+vx=6$ (3)

$vx^2=5$ (4)

From (3), $v=\frac{6-x}{x}$

From (4), $v=\frac{5}{x^2}$

Hence, $\frac{6-x}{x}=\frac{5}{x^2}$

Multiply each member by x^2, $6x-x^2=5$

$x^2-6x=-5$

$x^2-6x+9=4$

$x-3=\pm 2$

$x=5$ or 1

$y=1$ or 5

3. $\begin{cases} x^2+y^2=34. & (1) \\ xy=15. & (2) \end{cases}$

Add 2 times (2) to (1), $x^2+2xy+y^2=64$

Extract square root, $x+y=\pm 8$ (3)

Subtract 2 times (2) from (1), $x^2-2xy+y^2=4$

Extract square root, $x-y=\pm 2$

But (3) $x+y=\pm 8$

Therefore, adding, $2x=10$ or 6 or -10 or -6

Hence, $x=5$ or 3 or -5 or -3

Substituting in (2), $y=3$ or 5 or -3 or -5

4. $\begin{cases} x^3+y^3=72. & (1) \\ x+y=6. & (2) \end{cases}$

From (2), $y=6-x$

Cube, $y^3=216-108x+18x^2-x^3$

Substitute in (1), $216-108x+18x^2=72$

Transpose 216, $18x^2-108x=-144$

Divide by 18, $x^2-6x=-8$

Complete square, $x^2-6x+9=1$

Extract square root, $x-3=\pm 1$

Hence, $x=4$ or 2

and $y=2$ or 4

ANOTHER SOLUTION.

Divide (1) by (2), $x^2-xy+y^2=12$ (3)

Square (2), $x^2+2xy+y^2=36$ (4)

Subtract (3) from (4), $3xy=24$

or $xy=8$ (5)

Subtract (5) from (3), $x^2-2xy+y^2=4$

Extract square root, $x-y=\pm 2$

But (2) $x+y=6$

Adding, $2x=8$ or 4

Hence, $x=4$ or 2

and $y=2$ or 4

5. $\begin{cases} x^2+xy=35. & (1) \\ y^2+xy=14. & (2) \end{cases}$

Adding (1) and (2), $x^2+2xy+y^2=49$

Extract square root, $x-y=\pm 7$ (3)

Divide (1) by (3), $x=\pm 5$

Divide (2) by (3), $y=\pm 2$

ANOTHER SOLUTION.

Factor each left member, $x(x+y)=35$ (3)

$y(x+y)=14$ (4)

Divide (3) by (4), $\frac{x}{y}=\frac{35}{14}=\frac{5}{2}$

Multiply by y, $x=\frac{5}{2}y$

Substitute in (4), $\frac{7}{2}y^2=14$

Hence, $y^2=4$

Therefore, $y=\pm 2$

and $x=\pm 5$

6. $\begin{cases} x^4+y^4=272. & (1) \\ x+y=6. & (2) \end{cases}$

Raise (2) to the fourth power,

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=1296 \quad (3)$$

But (1) $x^4+y^4=272$

Subtract, $4x^3y+6x^2y^2+4xy^3=1024 \quad (4)$

Factor (4), $xy\,2x^2+3xy+2y^2)=512 \quad (5)$

Square (2) and multiply result by $2xy$,

$$xy\,2x^2+4xy+2y^2)=72xy \quad (6)$$

Take (5) from (6), $(xy)^2=72xy-512 \quad (7)$

Transpose, $(xy)^2-72xy=-512$

Complete square, $(xy)^2-72xy+1296=784$

Extract square root, $xy-36=\pm 28$

Hence, $xy=64$ or $8 \quad (8)$

Square (2), $x^2+2xy+y^2=36$

Multiply (8) by 4, $4xy=256$ or $32 \quad (9)$

Subtract, $x^2-2xy+y^2=-220$ or 4

Reject -220 because we cannot extract the square root of -220 and extract the square root of the resulting equations.

$x-y=\pm 2$

But (2) $x+y=6$

Adding, $2x=8$ or 4

Hence, $x=4$ or 2

and $y=2$ or 4

EXAMPLES.

Solve the following sets of equations:

7. $\begin{cases} x^3-y^3=63. \\ x-y=3. \end{cases}$

8. $\begin{cases} x^2+xy+y^2=91. \\ x^2-xy+y^2=31. \end{cases}$

9. $\begin{cases} xy+xy^2=12. \\ x+xy^3=13. \end{cases}$

10. $\begin{cases} x^4-y^4=609. \\ x-y=-4. \end{cases}$

11. $\begin{cases} x^3+y^3=7110. \\ x+y=30. \end{cases}$

12. $\begin{cases} \frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{2}. \\ x^2+y^2=8. \end{cases}$

13. $\begin{cases} x^2+y^2+x+y=18. \\ xy=6. \end{cases}$

14. $\begin{cases} x^2+y^2+x-y=78. \\ xy=24. \end{cases}$

15. $\begin{cases} x^3-y^3=19(x-y). \\ x^2-y^2=5. \end{cases}$

16. $\begin{cases} x^4+y^4=97. \\ x^2+y^2=13. \end{cases}$

17. $\begin{cases} \frac{1}{x}-\frac{1}{y}=\frac{1}{36}. \\ xy^2-x^2y=324. \end{cases}$

18. $\begin{cases} 16x^2=9y^2. \\ 2xy-5x+6y=33. \end{cases}$

19. $\begin{cases} x^2+xy=\frac{7}{3}(x+y). \\ y^2+xy=\frac{11}{3}(x+y). \end{cases}$

20. $\begin{cases} xy+x=20. \\ xy-y=12. \end{cases}$

21. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{9}{20}. \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{41}{400}. \end{cases}$

22. $\begin{cases} xy-x^2-y^2=-91. \\ x^2+y^2+xy=223. \end{cases}$

23. $\begin{cases} 42(x^2+y^2)=85xy. \\ 15xy=2520. \end{cases}$

24. $\begin{cases} x^2-x+138=2xy+3y \\ y^2+7y-3x=278. \end{cases}$

25. $\begin{cases} x^4+y^4=706. \\ x-y=2. \end{cases}$

26. $\begin{cases} x^3+y^3=152. \\ x^2-xy+y^2=19. \end{cases}$

27. $\begin{cases} \frac{x^2+y^2}{xy}=\frac{145}{72}. \\ xy^3+x^3y-x^2y^2=\frac{7\,3}{2\,8\,8}. \end{cases}$

28. $\begin{cases} x^3-y^3=279. \\ x^2+xy+y^2=93. \end{cases}$

29. $\begin{cases} x^3+y^3=133. \\ xy(x+y)=70. \end{cases}$

30. $\begin{cases} (x^2+y^2)(x+y)=272. \\ x^2+y^2+x+y=42. \end{cases}$

31. $\begin{cases} \frac{9}{x^2}-2y^2=-1. \\ \frac{2}{x^2}+9y^2=\frac{83}{9}. \end{cases}$

32. $\begin{cases} x^2(x+y)=\frac{2\,5\,6}{3}. \\ x^2(2x-2y)=\frac{2\,5\,6}{3}. \end{cases}$

EXERCISE 114.

PROBLEMS.

1. What two numbers are those whose product is 24 and whose sum added to the sum of their squares is 62?

2. Find two numbers such that the square of the greater minus the square of the less may be 56, and the square of the less plus $\frac{1}{3}$ of their product may be 40.

3. A number consists of two digits whose sum is 15. If 31 be added to the product of the digits the digits will be in the reverse order. What is the number?

4. There is a number consisting of two digits, which number divided by the sum of its digits gives a quotient 2 greater than the first digit. But if the digits be in the reverse order and the resulting number be divided by a number 1 greater than the sum of the digits, the quotient so obtained is greater by 2 than the preceding quotient. What is the number?

5. Find two numbers such that their product added to their sum gives 62, and their sum taken from the sum of their squares leaves a remainder of 86.

6. A certain number consists of two digits of which the digit in ten's place is 3 times the digit in unit's place, and if 12 be subtracted from the number itself the remainder will equal the square of the digit in ten's place. Find the number.

7. The sum of two numbers is 37 and the sum of their squares is greater by 9 than twice their product. What are the numbers?

8. The sum of two numbers is 22 and the sum of their cubes is 2926. What are the numbers?

9. The sum of the squares of two numbers is 410. If we diminish the greater by 4 and increase the less by 4 the sum of the squares of the two results is 394. What are the two numbers?

10. A man bought some horses for $1250. If he had bought 3 more and paid $25 less for each horse, they would have cost him $1300. How many horses did he buy, and at what price?

11. A stock dealer bought some horses and cows for $11600. The number of horses bought was equal to the number of dollars paid for each horse, and the number of cows bought was equal to the number of dollars for each cow. Had the number of horses bought been equal to the number dollars paid for each cow and the number of cows bought been equal to the number of dollars paid for each horse, the stock would have cost him $8000. How many horses and cows did he buy and what did he pay for each?

12. A lady bought 55 yards of cloth in two pieces, paying $17 altogether; for the first piece she paid twice as many cents per yard as there were yards in the piece, and for the second piece she paid one-half as many cents per yard as there were yards in the piece. What was the number of yards and the price per yard of each piece?

13. Find two numbers whose sum is 9 times their difference and whose product diminished by the greater is equal to 12 times the greater divided by the less.

14. Two rectangles contain the same area, 480 square yards. The difference of their lengths is 10 yards, and of their breadths is 4 yards; find their sides.

15. If a carriage wheel $14\frac{2}{3}$ feet in circumference take 1 second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less. How fast is the carriage traveling?

16. A cistern can be filled with water by two pipes. By one of these pipes alone the cistern would be filled 2 hours sooner than by the other; also, the cistern can be filled by both pipes together in $1\frac{7}{8}$ hours. Find the time each pipe alone would take to fill the cistern.

17. A man bought a number of shares of \$20 stock when they were at a certain rate per cent. discount for \$1500, and afterwards when they were at the same rate per cent. premium sold all but 60 shares for \$1000. How many shares did he buy, and what did he give for each of them?

18. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 yards; if the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards. Find the circumference of each wheel.

19. A sets off from London to York, and B at the same time from York to London, and each travels uniformly. A reaches York 16 hours and B reaches London 36 hours after they have met on the road. Find in what time each has performed the journey.

20. A man arrives at the railway station nearest his home $1\frac{1}{2}$ hours before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles per hour, and meeting his carriage when it had traveled 8 miles, reaches home one hour earlier than he had originally expected. How far is his home from the station, and at what rate was his carriage driven?

CHAPTER XVI.

THEORY OF INDICES.

EXERCISE 115.

MEANING OF FRACTIONAL EXPONENTS.

278. The exponents which we have considered heretofore are defined by the following equation:

$$a^n = aaaa \ . \ . \ . \text{ to } n \text{ factors.}$$

In other words, a^n is an abbreviated way of writing the product of n factors each equal to a.

279. It has been proved that exponents follow the five laws expressed by the following examples and formulas:

EXAMPLES.		FORMULAS.	
$a^{11} \times a^7 = a^{18}$	Art. 39,	$a^n \times a^r = a^{n+r}$	A
$a^{11} \div a^7 = a^4$	Art. 54,	$a^n \div a^r = a^{n-r}$ if $n > r$	B
$(a^{11})^7 = a^{77}$	Art. 134,	$(a^n)^r = a^{nr}$	C
$(abc)^7 = a^7 b^7 c^7$	Art. 135,	$(abc)^n = a^n b^n c^n$	D
$\left(\frac{a}{b}\right)^7 = \frac{a^7}{b^7}$	Art. 136,	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	E

We will find it convenient to refer to these formulas as A, B, C, D, and E, respectively, and we will speak of them collectively as **The Laws of Exponents or Indices.**

280. It is very plain that the exponent n must be a positive whole number in order that a^n may have a meaning by the definition in Art. 278. If, therefore, we wish to use symbols like $a^{\frac{3}{4}}$ or a^{-2} we must first find a meaning for such expressions.

* This symbol stands for the words "is greater than."

281. It is usual in Algebra to define the symbols which are to be used, and afterwards discover what laws these symbols follow in algebraic operations. Thus, in the first part of Algebra we defined positive integral exponents and subsequently proved that they follow the five laws, *A*, *B*, *C*, *D*, and *E*. But an exception to this general practice occurs in the case of fractional and negative exponents. *Here we first state some law which we wish fractional and negative exponents to follow and then seek what meaning must be given to these exponents in consequence of this law.* Thus, we say: *All fractional exponents must follow law A; required the meaning of fractional exponents.* We will consider a few special cases at first.

282. *Given that law A must hold for fractional exponents; required the meaning of* $a^{\frac{1}{2}}$.

Since law *A* must hold, we have

$$a^{\frac{1}{2}}a^{\frac{1}{2}}=a^{\frac{1}{2}+\frac{1}{2}}=a.$$

Thus $a^{\frac{1}{2}}$ must be such a number that if it be multiplied by itself the result is a. By definition the square root of a is such a number; therefore $a^{\frac{1}{2}}$ must be equivalent to the square root of a. Thus we say

$$a^{\frac{1}{2}}=\sqrt{a}.$$

283. *Given that law A must hold for fractional exponents; required the meaning of* $a^{\frac{1}{3}}$.

Since law *A* must hold, we have

$$a^{\frac{1}{3}}a^{\frac{1}{3}}a^{\frac{1}{3}}=a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=a.$$

But, by definition of a cube root,

$$\sqrt[3]{a}\sqrt[3]{a}\sqrt[3]{a}=a.$$

Therefore, $a^{\frac{1}{3}}$ must be equivalent to the cube root of a; that is,

$$a^{\frac{1}{3}}=\sqrt[3]{a}.$$

284. *Given that law A must hold for fractional exponents; required the meaning of* $a^{\frac{3}{4}}$.

Since law A must hold, we have

$$a^{\frac{3}{4}}a^{\frac{3}{4}}a^{\frac{3}{4}}a^{\frac{3}{4}}=a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}}=a^3.$$

Thus we see that $a^{\frac{3}{4}}$ is one of the four equal factors which, when multiplied together, produce a^3. Hence, by the definition of a root, $a^{\frac{3}{4}}$ must be equivalent to the fourth root of a^3; that is,

$$a^{\frac{3}{4}}=\sqrt[4]{a^3}.$$

285. *Given that law A must hold for fractional exponents; required the meaning of* $a^{\frac{1}{r}}$.

Let r be any positive whole number. Since law A must hold, we have

$$a^{\frac{1}{r}}a^{\frac{1}{r}}a^{\frac{1}{r}} \ldots \text{ to } r \text{ factors}=a^{\frac{1}{r}+\frac{1}{r}+\frac{1}{r}+\ldots \text{ to } r \text{ terms}}=a.$$

Thus we see that $a^{\frac{1}{r}}$ is one of the r equal factors which, when multiplied together, produce a. Hence, by the definition of a root, $a^{\frac{1}{r}}$ *is equivalent to the* r*th root of* a; that is,

$$a^{\frac{1}{r}}=\sqrt[r]{a}. \qquad [1]$$

286. *Given that law A must hold for fractional exponents; required the meaning of* $a^{\frac{n}{r}}$.

It is here supposed that n and r stand for any positive whole numbers, that is, that $\frac{n}{r}$ is any positive fraction. Since law A must hold, we have

$$a^{\frac{n}{r}}a^{\frac{n}{r}}a^{\frac{n}{r}} \ldots \text{ to } r \text{ factors}=a^{\frac{n}{r}+\frac{n}{r}+\frac{n}{r}+\ldots \text{ to } r \text{ terms}}=a^n.$$

Thus we see that $a^{\frac{n}{r}}$ is one of the r equal factors which, when multiplied together, produce a^n. Hence, by the definition of a root, $a^{\frac{n}{r}}$ *is equivalent to the* r*th root of the* n*th power of* a; that is,

$$a^{\frac{n}{r}}=\sqrt[r]{a^n}. \qquad [2]$$

287. Thus, by requiring that fractional exponents shall follow law A, we have found that

Any positive fractional exponent indicates a root of a power, where the numerator shows the power and the denominator the root.

288. We will now get at the meaning of a fractional exponent by a little different process, but still by requiring that such exponents shall follow law A.

From the meaning of a positive integral exponent (Art. 278) we know

$$(a^{\frac{1}{r}})^n = a^{\frac{1}{r}} a^{\frac{1}{r}} a^{\frac{1}{r}} \ldots \text{ to } n \text{ factors.}$$

By law A, $$= a^{\frac{1}{r}+\frac{1}{r}+\frac{1}{r}+\ldots \text{ to } n \text{ terms}}.$$

Adding the fractions, $$= a^{\frac{n}{r}}.$$

Therefore we have shown that

$$(a^{\frac{1}{r}})^n = a^{\frac{n}{r}}, \qquad [3]$$

or, since $a^{\frac{1}{r}} = \sqrt[r]{a}$, by [1], we have shown that

$$a^{\frac{n}{r}} = (\sqrt[r]{a})^n; \qquad [3]$$

That is, $a^{\frac{n}{r}}$ *is equivalent to the n th power of the r th root of a.*

289. Thus we have shown that

Any positive fractional exponent indicates a power of a root, where the numerator shows the power and the denominator the root.

290. Comparing this with Art. 287, we see we have found *two meanings* for $a^{\frac{n}{r}}$; first, the r th root of the n th power of a; second, the n th power of the r th root of a. From equations [2] and [3] we get this statement in the form of an equation as follows:

$$a^{\frac{n}{r}} = \sqrt[r]{a^n} = (\sqrt[r]{a})^n, \qquad [4]$$

or, writing exactly the same equation, but using fractional exponents instead of radical signs,

$$a^{\frac{n}{r}} = (a^n)^{\frac{1}{r}} = (a^{\frac{1}{r}})^n. \qquad [4]$$

While we have found *two meanings* for $a^{\frac{n}{r}}$ yet these two meanings always give *the same numerical results*, as is proved by equation [4]. Thus: $8^{\frac{2}{3}}=\sqrt[3]{8^2}=(\sqrt[3]{8})^2$, $81^{\frac{2}{4}}=\sqrt[4]{81^2}=(\sqrt[4]{81})^2$, etc. The order of the operations is different, but the results are the same.

291. The above truth is very important and may be stated in the following manner:

The r th root of the n th power of a number is equal to the n th power of the r th root of that number.

EXERCISE 116.

EXAMPLES.

Write each of the following sixteen expressions, using fractional exponents in place of radical signs:

1. $\sqrt{a}$.	5. $\sqrt[7]{a^5}$.	9. $\sqrt[5]{x^3}$.	13. $\sqrt[5]{a-5}$.
2. $\sqrt{a^3}$.	6. $(\sqrt{a})^6$.	10. $(\sqrt[5]{x})^3$.	14. $(\sqrt[5]{x-y})^4$.
3. $\sqrt[3]{a^2}$.	7. $\sqrt[7]{a^5}$.	11. $\sqrt[5]{x^4}$.	15. $\sqrt{a^2-b^2}$.
4. $\sqrt[5]{a^3}$.	8. $(\sqrt[4]{a})^5$.	12. $(\sqrt[5]{x})^4$.	16. $\sqrt[5]{(a+x)^4}$.

Find the numerical value of each of the following sixteen expressions:

17. $4^{\frac{1}{2}}$.	21. $625^{\frac{1}{4}}$.	25. $81^{\frac{3}{2}}$.	29. $256^{\frac{3}{4}}$.
18. $27^{\frac{1}{3}}$.	22. $64^{\frac{1}{6}}$.	26. $125^{\frac{5}{3}}$.	30. $64^{\frac{5}{6}}$.
19. $9^{\frac{1}{2}}$.	23. $216^{\frac{1}{3}}$.	27. $32^{\frac{4}{5}}$.	31. $512^{\frac{2}{3}}$.
20. $16^{\frac{1}{4}}$.	24. $16^{\frac{5}{4}}$.	28. $81^{\frac{5}{4}}$.	32. $128^{\frac{4}{7}}$.

Write each of the following expressions *in two ways*, using radical signs instead of fractional exponents:

33. $a^{\frac{1}{3}}$.	37. $n^{\frac{2}{3}}$.	41. $r^{\frac{7}{8}}$.	45. $a^{\frac{n}{3}}$.
34. $1^{\frac{2}{5}}$.	38. $b^{\frac{3}{4}}$.	42. $x^{\frac{2}{q}}$.	46. $b^{\frac{r}{2n}}$.
35. $m^{\frac{3}{4}}$.	39. $e^{\frac{5}{6}}$.	43. $y^{\frac{1}{t}}$.	47. $x^{\frac{n+1}{r}}$.
36. $x^{\frac{3}{7}}$.	40. $h^{\frac{1}{9}}$.	44. $d^{\frac{x}{y}}$.	48. $a^{\frac{r+s}{t}}$.

EXERCISE 117.

PROPERTIES OF FRACTIONAL EXPONENTS.

292. *To prove that a number with a fractional exponent has the same value whether the exponent is expressed in its lowest terms or not.*

Let n, r, and t be any positive whole numbers. We are to prove that $a^{\frac{n}{r}} = a^{\frac{nt}{rt}}$.

We know that $a = (a^{\frac{1}{rt}})^{rt}$

because the rtth power of the rtth root equals the number itself. Taking the rth root of each side,

$$a^{\frac{1}{r}} = (a^{\frac{1}{rt}})^{t}.$$

Raising both sides to the nth power,

$$(a^{\frac{1}{r}})^{n} = (a^{\frac{1}{rt}})^{nt}.$$

Each side is now a power of a root. By the meaning of fractional indices (Art. 289) we write this

$$a^{\frac{n}{r}} = a^{\frac{nt}{rt}}, \qquad [5]$$

which is what was to be proved.

293. Having found a meaning for fractional exponents by requiring that they follow law A, we will now prove that they also follow laws B, C, D, and E.

294. *To prove that fractional exponents follow law B.*

Let n, r, s, and t be any positive whole numbers, so that $\frac{n}{r}$ and $\frac{s}{t}$ are any positive fractions. We will prove

$$a^{\frac{n}{r}} \div a^{\frac{s}{t}} = a^{\frac{n}{r} - \frac{s}{t}}, \text{ if } \tfrac{n}{r} > \tfrac{s}{t}.$$

We know $a^{\frac{n}{r}} \div a^{\frac{s}{t}} = a^{\frac{nt}{rt}} \div a^{\frac{sr}{tr}}$, by Art. 292.

$= (a^{\frac{1}{rt}})^{nt} \div (a^{\frac{1}{rt}})^{sr}$, by equation [4].

$= (a^{\frac{1}{rt}})^{nt-sr}$, by law B for integral indices, since nt and sr are whole numbers and $nt > sr$ if $\frac{n}{r} > \frac{s}{t}$.

$$= a^{\frac{nt-sr}{rt}}, \qquad \text{by Art. 289.}$$

But this last fractional exponent is what we would get if we subtract $\frac{s}{t}$ from $\frac{n}{r}$.

Therefore, $a^{\frac{n}{r}} \div a^{\frac{s}{t}} = a^{\frac{n}{r}-\frac{s}{t}}$, if $\frac{n}{r} > \frac{s}{t}$, [6]

which is what was to be proved.

295. *To prove that fractional exponents follow law C.*

Let n, r, s, and t be any positive whole numbers.

CASE I. We will prove $(a^{\frac{n}{r}})^s = a^{\frac{ns}{r}}$.

We know $(a^{\frac{n}{r}})^s = a^{\frac{n}{r}} a^{\frac{n}{r}} a^{\frac{n}{r}} \ldots$ to s factors,

by definition of an integral index.

$= a^{\frac{n}{r}+\frac{n}{r}+\frac{n}{r}+\ldots \text{ to } s \text{ terms}}$, by law *A*.

$= a^{\frac{ns}{r}}$, by adding fractional indices.

Therefore, $(a^{\frac{n}{r}})^s = a^{\frac{ns}{r}}$. [7]

which is what was to be proved.

CASE II. We will prove $(a^{\frac{n}{r}})^{\frac{1}{t}} = a^{\frac{n}{rt}}$,

We know $(a^{\frac{n}{r}})^{\frac{1}{t}} = (a^{\frac{nt}{rt}})^{\frac{1}{t}}$, by Art. 292.

$= ([a^{\frac{1}{rt}}]^{nt})^{\frac{1}{t}}$,

by meaning of a fractional index.

$= (\{[a^{\frac{1}{rt}}]^n\}^t)^{\frac{1}{t}}$,

by law *C* for integral indices.

$= [a^{\frac{1}{rt}}]^n$,

by taking the tth root of the tth power.

$= a^{\frac{n}{rt}}$,

by meaning of fractional index.

Therefore, $(a^{\frac{n}{r}})^{\frac{1}{t}} = a^{\frac{n}{rt}}$, [8]

which is what was to be proved.

CASE III. We will prove $(a^{\frac{n}{r}})^{\frac{s}{t}}=a^{\frac{ns}{rt}}$.

We know $(a^{\frac{n}{r}})^{\frac{s}{t}}=[(a^{\frac{n}{r}})^{\frac{1}{t}}]^s$,

by the meaning of a fractional index.

$=[a^{\frac{n}{rt}}]^s$, by Case II.

$=a^{\frac{ns}{rt}}$, by Case I.

Therefore, $(a^{\frac{n}{r}})^{\frac{s}{t}}=a^{\frac{ns}{rt}}$, [9]

which is what was to be proved.

296. *To prove that fractional exponents follow law D.*

We are to prove $(abc)^{\frac{n}{r}}=a^{\frac{n}{r}}b^{\frac{n}{r}}c^{\frac{n}{r}}$.

We know $(abc)^{\frac{n}{r}}=[(abc)^n]^{\frac{1}{r}}$,

by meaning of a fractional index.

$=[a^nb^nc^n]^{\frac{1}{r}}$,

by law *D* for integral indices.

$=[(a^{\frac{n}{r}})^r(a^{\frac{n}{r}})^r(a^{\frac{n}{r}})^r]^{\frac{1}{r}}$,

by law *C*, (Art. 295, Case I).

$=[(a^{\frac{n}{r}}b^{\frac{n}{r}}c^{\frac{n}{r}})^r]^{\frac{1}{r}}$,

by law *D* for integral indices.

$=a^{\frac{n}{r}}b^{\frac{n}{r}}c^{\frac{n}{r}}$,

by taking the rth root of the rth power.

Therefore, $(abc)^{\frac{n}{r}}=a^{\frac{n}{r}}b^{\frac{n}{r}}c^{\frac{n}{r}}$, [10]

which is what was to be proved.

297. *To prove that fractional exponents follow law E.*

We are to prove $\left(\frac{a}{b}\right)^{\frac{n}{r}}=\frac{a^{\frac{n}{r}}}{b^{\frac{n}{r}}}$.

We know $\left(\frac{a}{b}\right)^{\frac{n}{r}}=\left[\left(\frac{a}{b}\right)^n\right]^{\frac{1}{r}}$

by the meaning of a fractional index.

$$= \left(\frac{a^n}{b^n}\right)^{\frac{1}{r}}$$

by law E for integral indices.

$$= \frac{(a^{\frac{n}{r}})^r}{(b^{\frac{n}{r}})^r}$$

by law C, (Art. 295, Case I).

$$= \left[\left(\frac{a^{\frac{n}{r}}}{a^{\frac{n}{r}}}\right)^r\right]^{\frac{1}{r}}$$

by law E for integral indices.

$$= \frac{a^{\frac{n}{r}}}{b^{\frac{n}{r}}}$$

by taking rth root of rth power.

Therefore, $$\left(\frac{a}{b}\right)^{\frac{n}{r}} = \frac{a^{\frac{n}{r}}}{b^{\frac{n}{r}}} \qquad [11]$$

which is what was to be proved.

EXERCISE 118.

EXAMPLES.

Perform the indicated operations in each of the following examples by means of the laws of fractional exponents just proved:

1. $a^{\frac{1}{5}} \times a^{\frac{3}{5}}$.

$$a^{\frac{1}{5}} \times a^{\frac{3}{5}} = a^{\frac{1}{5}+\frac{3}{5}} (\text{by } A) = a^{\frac{4}{5}}.$$

2. $a^{\frac{2}{3}} \times a^{\frac{3}{4}}$.

$$a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} (\text{by } A) = a^{1\frac{8}{12}+\frac{9}{12}} = a^{1\frac{17}{12}}.$$

3. $a^{\frac{1}{3}} \times a^{\frac{2}{5}}$.
4. $x^{\frac{5}{4}} \times x^{\frac{5}{2}}$.
5. $x^{\frac{2}{3}} \times x^{\frac{3}{2}}$.
6. $c^{\frac{1}{5}} \times c^{\frac{3}{10}}$.
7. $d^{\frac{4}{5}} \times d$.
8. $m^{\frac{2}{3}} \times m^3$.
9. $x^{\frac{1}{4}} \times x^{\frac{1}{5}}$.
10. $b^{\frac{3}{7}} \times b^{\frac{2}{3}}$.
11. $a^2 b^{\frac{1}{3}} \times 2a^3 b^{\frac{2}{3}}$.
12. $3x^{\frac{1}{2}} y^4 \times 2x^{\frac{3}{2}} y^{\frac{5}{4}}$
13. $a^{\frac{1}{2n}} \times a^{\frac{4}{3n}}$.
14. $a^{\frac{2}{n}} \times a^{\frac{r}{2n}}$.

15. $a^{\frac{3}{4}} \div a^{\frac{1}{4}}$.

$a^{\frac{3}{4}} \div a^{\frac{1}{4}} = a^{\frac{3}{4}-\frac{1}{4}}$(by B)$= a^{\frac{2}{4}} = a^{\frac{1}{2}}$(by Art. 292).

16. $b^{\frac{5}{7}} \div b^{\frac{2}{3}}$.

$b^{\frac{5}{7}} \div b^{\frac{2}{3}} = b^{\frac{5}{7}-\frac{2}{3}}$(by B)$= b^{\frac{15}{21}-\frac{14}{21}} = b^{\frac{1}{21}}$.

17. $a^{\frac{1}{2}} \div a^{\frac{1}{4}}$.

18. $h^{\frac{2}{3}} \div h^{\frac{1}{3}}$.

19. $k^{\frac{3}{4}} \div k^{\frac{1}{2}}$.

20. $l^{\frac{3}{5}} \div l^{\frac{3}{10}}$.

21. $x^{\frac{7}{3}} \div x^{2}$.

22. $6x^{\frac{2}{3}} \div 3x^{\frac{1}{2}}$.

23. $m^{\frac{1}{4}} \div m^{\frac{1}{5}}$.

24. $9a^{\frac{3}{5}} \div a^{\frac{1}{2}}$.

25. $a^{\frac{3n}{r}} \div a^{\frac{n}{2r}}$.

26. $8a^5 b^{\frac{3}{2}} \div 4a^2 b^{\frac{1}{2}}$.

27. $6x^6 y^{\frac{1}{3}} \div 3x^{\frac{7}{2}} y^{\frac{1}{4}}$.

28. $ab^{\frac{n}{r}} \div a^{\frac{1}{2}} b^{\frac{n}{4r}}$.

---

29. $(a^{\frac{1}{3}})^{\frac{1}{4}}$.

$(a^{\frac{1}{3}})^{\frac{1}{4}} = a^{\frac{1}{12}}$(by C).

30. $(b^{\frac{4}{5}})^{\frac{7}{8}}$.

$(b^{\frac{4}{5}})^{\frac{7}{8}} = b^{\frac{28}{40}}$(by C)$= b^{\frac{7}{10}}$(by Art. 292).

31. $(x^{\frac{2}{3}})^{\frac{3}{4}}$.

32. $(a^{\frac{2}{3}})^{\frac{6}{10}}$.

33. $(a^{\frac{1}{2}})^{\frac{4}{5}}$.

34. $(h^{\frac{2}{5}})^{\frac{5}{6}}$.

35. $(y^{\frac{2}{3}})^{\frac{3}{2}}$.

36. $(a^{\frac{3}{5}})^{\frac{5}{6}}$.

37. $(a^{\frac{3}{2}})^{4}$.

38. $(a^{\frac{5}{12}})^{4}$.

39. $(x^{\frac{n}{2}})^{\frac{6}{5n}}$.

40. $\left[(x^{n})^{\frac{1}{n}}\right]^{\frac{n}{r}}$.

41. $\left[(x^{\frac{1}{n}})^{r}\right]^{n}$.

42. $(x^{\frac{st}{6rn}})^{\frac{2}{t}}$.

---

43. $(abc)^{\frac{1}{3}}$.

$(abc)^{\frac{1}{3}} = a^{\frac{1}{3}} b^{\frac{1}{3}} c^{\frac{1}{3}}$(by D).

44. $(a^4 x^2 y^6)^{\frac{3}{2}}$.

$(a^4 x^2 y^6)^{\frac{3}{2}} = (a^4)^{\frac{3}{2}} (x^2)^{\frac{3}{2}} (y^6)^{\frac{3}{2}}$(by D)$= a^6 x^3 y^9$(by C).

45. $(a^{\frac{2}{3}} b^{\frac{1}{2}} c^{\frac{3}{4}})^{\frac{6}{5}}$.

$(a^{\frac{2}{3}} b^{\frac{1}{2}} c^{\frac{3}{4}})^{\frac{6}{5}} = (a^{\frac{2}{3}})^{\frac{6}{5}} (b^{\frac{1}{2}})^{\frac{6}{5}} (c^{\frac{3}{4}})^{\frac{6}{5}}$(by D)$= a^{\frac{4}{5}} b^{\frac{3}{5}} c^{\frac{9}{10}}$(by C).

46. $(a^2 b^3)^{\frac{1}{6}}$.

47. $(a^{\frac{3}{4}} b^{\frac{6}{5}})^{\frac{10}{3}}$.

48. $(a^2 d^{\frac{1}{2}})^{\frac{1}{2}}$.

49. $(a^5 b^{\frac{1}{2}})^{\frac{2}{5}}$.

50. $(x^{12} y^6)^{\frac{5}{6}}$.

51. $(2^{\frac{1}{2}} x^{\frac{5}{6}} y^{\frac{3}{4}})^{12}$.

52. $(32x^{\frac{3}{8}} y^{\frac{9}{4}})^{\frac{4}{9}}$.

53. $(a^{\frac{1}{2}} x^{\frac{1}{7}} y^{\frac{1}{8}})^{14}$.

54. $(a^{\frac{2}{3}} b^{\frac{3}{4}} c^{\frac{4}{5}})^{\frac{5}{6}}$.

55. $(8a^6 b^{\frac{3}{2}} c^{\frac{3}{4}})^{\frac{2}{3}}$.

56. $(36a^4 x^2 y^3)^{\frac{1}{2}}$.

57. $(\frac{1}{8} a^5 b^3 c^2)^{\frac{1}{3}}$.

58. $\left(\frac{a^{\frac{2}{3}}}{b^{\frac{3}{4}}}\right)^{\frac{1}{6}}$.

$$\left(\frac{a^{\frac{2}{3}}}{b^{\frac{3}{4}}}\right)^{\frac{1}{6}}=\frac{(a^{\frac{2}{3}})^{\frac{1}{6}}}{(b^{\frac{3}{4}})^{\frac{1}{6}}}(\text{by } E)=\frac{a^{\frac{1}{9}}}{b^{\frac{1}{8}}}(\text{by } C).$$

59. $\left(\frac{a}{b}\right)^{\frac{1}{2}}$.

60. $\left(\frac{x}{y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$.

61. $\left(\frac{a^2}{b^3}\right)^{\frac{1}{5}}$.

62. $\left(\frac{x}{y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$.

63. $\left(\frac{a^3}{b^{\frac{2}{3}}}\right)^{\frac{2}{7}}$.

64. $\left(\frac{a^5}{b^{\frac{2}{3}}}\right)^{\frac{3}{5}}$.

65. $\left(\frac{8x^{\frac{1}{2}}}{y^{\frac{2}{3}}}\right)^{\frac{2}{3}}$.

66. $\left(\frac{a^{\frac{n}{r}}}{b^{\frac{s}{t}}}\right)^{\frac{t}{q}}$.

67. $\left(\frac{9a^{\frac{3}{4}}}{4\,b^{\frac{2}{5}}}\right)^{\frac{3}{2}}$.

---

68. $(a^{\frac{4}{3}}+a^{\frac{2}{3}}+1)(a^{\frac{5}{3}}+a-a^{\frac{1}{3}})$.

We arrange the work thus:

$$\begin{array}{l}
a^{\frac{4}{3}}+a^{\frac{2}{3}}+1 \\
a^{\frac{5}{3}}+a\ -a^{\frac{1}{3}} \\
\hline
a^3+a^{\frac{7}{3}}+a^{\frac{5}{3}} \\
\qquad a^{\frac{7}{3}}+a^{\frac{5}{3}}+a \\
\qquad\qquad -a^{\frac{5}{3}}-a-a^{\frac{1}{3}} \\
\hline
a^3+2a^{\frac{7}{3}}+a^{\frac{5}{3}}\qquad -a^{\frac{1}{3}}
\end{array}$$

69. $(x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y)(x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y)$.

70. $(x+2y^{\frac{1}{2}}+3y^{\frac{1}{3}})(x-2y^{\frac{1}{2}}+3y^{\frac{1}{3}})$.

71. $(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})(a^{\frac{1}{3}}-b^{\frac{1}{3}})$.

72. $(x^{\frac{3}{4}}+y^{\frac{3}{4}})(x^{\frac{3}{4}}-y^{\frac{3}{4}})$.

73. $(x^{\frac{3}{2}}-x+x^{\frac{1}{2}}-1)(x+x^{\frac{1}{2}}+1)$.

74. $(a^{\frac{4}{5}}-3a^{\frac{3}{5}}b^{\frac{1}{2}}+4a^{\frac{2}{5}}b-a^{\frac{1}{5}}b^{\frac{3}{2}})(a^{\frac{2}{5}}-2a^{\frac{1}{5}}b^{\frac{1}{2}})$.

75. $(x^{\frac{n}{2}}-x^{\frac{n}{4}}y^{\frac{r}{4}}+y^{\frac{r}{2}})(x^{\frac{n}{2}}+x^{\frac{n}{4}}y^{\frac{r}{4}}+y^{\frac{r}{2}})$.

76. $(a^{\frac{3}{p}}-2a^{\frac{2}{p}}+3a^{\frac{1}{p}})(2a^{\frac{2}{p}}-a^{\frac{1}{p}})$.

77. $(2ab^{\frac{2}{3}}-3a^{\frac{3}{5}}b^{\frac{4}{5}})^2$.

78. $(x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}})\div(x^{\frac{1}{2}}-y^{\frac{1}{2}})$.

We arrange the work as follows:

$$
\begin{array}{r|l}
x^{\frac{1}{2}}-y^{\frac{1}{2}} & x^{\frac{3}{2}}-xy^{\frac{1}{2}}+x^{\frac{1}{2}}y-y^{\frac{3}{2}} \;(\; x+y \\
\cline{2-2}
 & x^{\frac{3}{2}}-xy^{\frac{1}{2}} \\
\cline{2-2}
 & x^{\frac{1}{2}}y-y^{\frac{3}{2}} \\
 & x^{\frac{1}{2}}y-y^{\frac{3}{2}} \\
\cline{2-2}
\end{array}
$$

It is just as important to keep dividend and divisor arranged according to the powers of some letter in case the exponents are fractional as in the case they are integral. The fractional and integral exponents must take the order of their respective magnitudes.

79. $(x^{\frac{5}{2}}-x^2-4x^{\frac{3}{2}}+6x-2x^{\frac{1}{2}})\div(x^{\frac{3}{2}}-4x^{\frac{1}{2}}+2)$.

80. $(x-1)\div(x^{\frac{1}{3}}-1)$.

81. $(y-1)\div(y^{\frac{1}{6}}-1)$.

82. $(a-b^2)\div(a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{\frac{1}{4}}b+b^{\frac{3}{2}})$.

83. $(a^{\frac{3}{4}}-2a^{\frac{3}{8}}x^{\frac{3}{2}}+x^3)\div(a^{\frac{1}{4}}-2a^{\frac{1}{8}}x^{\frac{1}{2}}+x)$.

84. $(x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}})\div(x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}})$.

85. $(a^{\frac{1}{2}}-b^{\frac{1}{2}}-c^{\frac{1}{2}}+2b^{\frac{1}{4}}c^{\frac{1}{4}})\div(a^{\frac{1}{4}}+b^{\frac{1}{4}}-c^{\frac{1}{4}})$.

86. $(8x^{\frac{3}{2}}+y^{\frac{3}{2}}-z+6x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{3}})\div(2x^{\frac{1}{2}}+y^{\frac{1}{2}}-z^{\frac{1}{3}})$.

87. Find square root of $x^{\frac{2}{3}}+2x^{\frac{1}{3}}+1$.

88. Find square root of $4x^{\frac{2}{3}}-4x^{\frac{1}{3}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

89. Factor $x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

90. Find square root of $x^{\frac{2}{3}}-4x^{\frac{5}{6}}+4x+2x^{\frac{7}{6}}-4x^{\frac{4}{3}}+x^{\frac{5}{3}}$.

EXERCISE 119.

MEANING OF ZERO AND NEGATIVE EXPONENTS.

298. If the product of two numbers is unity, either of the numbers is called the **Reciprocal** of the other number. Thus, $\frac{1}{2}$ is the reciprocal of 2, $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, etc. In other words, the reciprocal of a number is 1 divided by that number.

299. *Given that negative exponents must follow law A; required the meaning of* a^{-2}.

Since law A must hold, we have

$$a^3 a^{-2} = a^{3-2} = a.$$

Therefore, by dividing both sides by a^3,

$$a^{-2} = \frac{a}{a^3} = \frac{1}{a^2};$$

That is,

$$a^{-2} = \frac{1}{a^2}.$$

Hence, a^{-2} *is equivalent to the reciprocal of* a^2.

300. *Given that negative exponents must follow law A; required the meaning of* $a^{-\frac{2}{3}}$.

Since law A must hold, we have

$$a\, a^{-\frac{2}{3}} = a^{1-\frac{2}{3}} = a^{\frac{1}{3}}.$$

Therefore,

$$a^{-\frac{2}{3}} = \frac{a^{\frac{1}{3}}}{a} = \frac{1}{a^{\frac{2}{3}}};$$

That is,

$$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}}.$$

Hence, $a^{-\frac{2}{3}}$ *is equivalent to the reciprocal of* $a^{\frac{2}{3}}$.

301. *Given that negative exponents must follow law A; required the meaning of* a^{-n}.

Let n be any positive number, *integral or fractional*, so that $-n$ stands for a negative whole number or fraction. Since law A must hold, we have

$$a^r a^{-n} = a^{r-n}.$$

Let r be taken greater than n. Then we know

$$\frac{a^r}{a^n} = a^{r-n}.$$

Therefore, $$a^r a^{-n} = \frac{a^r}{a^n}.$$

Dividing both sides by a^r, we obtain

$$a^{-n} = \frac{1}{a^n}. \qquad [12]$$

Hence, a^{-n} *is equivalent to the reciprocal of* a^n.

302. Since n stands for any whole number or fraction in the above work, we may say:

Any number with a negative exponent is equivalent to the reciprocal of that number with the same exponent taken positive.

303. *Given that zero exponents must follow law A; required the meaning of* a^0.

Since law A must hold, we have

$$a^n a^0 = a^{n+0} = a^n.$$

Therefore, $$a^0 = 1.$$

That is, a^0 *is equivalent to unity.*

304. Since a in the above work stands for any number whatever, positive or negative, integral or fractional, we may say:

Any number with the exponent zero is equivalent to unity.

Thus, $2^0 = 1$, $3^0 = 1$, $5^0 = 1$ $10^0 = 1$, $x^0 = 1$, $(xy)^0 = 1$, $(2a + 4x^2)^0 = 1$, etc.

305. It follows necessarily from Art. 301 that

$$a^n b^{-r} c^s d^{-t} = \frac{a^n b^{-r} c^s}{d^t} = \frac{a^n c^s}{b^r d^t} = \frac{a^n}{c^{-s} b^r d^t} = \frac{1}{a^{-n} b^r c^{-s} d^t} = \frac{b^{-r} d^{-t}}{a^{-n} c^{-s}} \text{ etc.}$$

That is: *Any* FACTOR *may be transferred from one term of a fraction to the other term provided the sign of the exponent of that factor be changed.*

Thus, $\frac{2^{-3}x}{3ab^{-2}} = \frac{b^2 x}{24a}$; $\frac{4^{-\frac{1}{2}}a^{-1}b^2}{8^{-\frac{1}{3}}x^2 b^{-2}} = \frac{8^{\frac{1}{3}}b^2 b^2}{4^{\frac{1}{2}}a x^2}$, etc., etc.

EXERCISE 120.

EXAMPLES.

Find the numerical value of each of the following:

1. 2^0.
2. 2^{-1}.
3. 3^{-4}.
4. $(-2)^{-3}$.
5. 10^{-5}.
6. 1^{-1}.
7. 1^{-n}.
8. $8a^0 2^{-2}$.
9. $2(a+b)^0$.
10. $16^{-\frac{3}{4}}$.
11. $64^{-\frac{2}{3}}$.
12. $81^{-\frac{1}{2}}$.
13. $1024^{-\frac{3}{5}}$.
14. $512^{-\frac{1}{3}}$.
15. $64^{-\frac{5}{6}}$.
16. $625^{-\frac{3}{4}}$.

17. $\frac{1}{2^0}$.
18. $\frac{1}{1^{-1}}$.
19. $\frac{2}{3^{-2}}$.
20. $\frac{5}{(-4)^{-3}}$.
21. $\frac{2^{-4}}{4^{-2}}$.
22. $\frac{1^{-8}}{8^{-1}}$.
23. $\frac{5^{-2}}{36^{-\frac{1}{2}}}$.
24. $\frac{27^{-\frac{1}{3}}}{2^{-3}}$.
25. $\frac{32^{-\frac{1}{5}}}{2^{-1}}$.
26. $\frac{16^{-\frac{3}{4}}}{5^0}$.
27. $\frac{9^{-2}}{81^{-\frac{5}{4}}}$.
28. $\frac{7^0}{49^{-\frac{1}{2}}}$.

Write each of the following expressions without using zero or negative exponents:

29. x^0.
30. a^{-3}.
31. $x^2 y^{-2}$.
32. $5a^{-5}$.
33. $6a^2 b^{-3}$.
34. $3a^{-2} b^{-\frac{1}{2}}$.
35. $(x+y)^0$.
36. $x^0 + y^0$.
37. $(-x)^{-3}$.
38. $2a^3 x^{-2} y^{-\frac{1}{2}}$.
39. $a^0 b^{-\frac{2}{3}} c^{-2} d^3$.
40. $(-a^2)^{-3}$.

41. $\frac{1}{a^{-4}}$. 45. $\frac{n^{-3}}{r^{-5}}$. 49. $\left(\frac{x}{y}\right)^0$. 53. $\frac{a^{-\frac{1}{2}}}{b^{-2}c^2}$.

42. $\frac{1}{x^0}$. 46. $\frac{2a^{-2}}{3b^2x^{-3}}$. 50. $\frac{x^0}{y^0}$. 54. $\frac{a^{-\frac{2}{3}}b^{\frac{1}{6}}}{3x^{-\frac{1}{2}}b^{-\frac{1}{6}}}$.

43. $\frac{m^{-3}}{x^0}$. 47. $\frac{a^2b^{-4}}{x^{-3}y^{-5}}$. 51. $\frac{3a^{\frac{2}{3}}b^{-1}}{5a^{-\frac{2}{3}}b}$. 55. $\frac{3a^0b^{-2}c^{-4}}{5a^{-1}b^{-3}c^{-5}}$.

44. $\frac{a^2}{a^{-3}}$. 48. $\frac{a^{-2}b^4}{x^5y^3}$. 52. $\frac{(1-x)^{-1}}{y^{-7}}$ 56. $\frac{2a^0x^{-\frac{2}{5}}y^{-1}}{4^{-2}a^{-4}x^{\frac{3}{5}}y^{-3}}$

Write each of the following expressions *in one line*:

57. $\frac{1}{a^2}$. 60. $\frac{2x^{-1}y^5}{3a^{-2}y^{-3}}$. 63. $\frac{x^{-4}y^2}{5a^nb^{-r}}$.

58. $\frac{4}{a^r}$. 61. $\frac{3x^3y^{-3}}{4r^{-2}t^5}$. 64. $\frac{x^2y^3(a-b)^0}{ax^{-5}(a-b)^{-2}}$.

59. $\frac{ab^2}{c^3d^4}$. 62. $\frac{4x^{-5}y^{-6}}{u^4z^{-7}}$. 65. $\frac{3(a^0+b^0)^{-2}x^2}{5x^{\frac{1}{2}}y^{-\frac{2}{3}}}$

66. $\frac{ab^{\frac{1}{2}}}{x^{-\frac{1}{3}}y^{\frac{s}{t}}}$. 68. $\frac{a}{x^ny^{-2}z^0u^{\frac{p}{q}}w^{-\frac{s}{t}}}$.

67. $\frac{16(a+b)^{-1}c^{\frac{1}{2}}}{(a-b)^{-\frac{3}{2}}c^{-\frac{5}{2}}}$. 69. $\frac{a}{x^3}+\frac{b}{x^2}+\frac{c}{x}+\frac{d}{x^0}$.

EXERCISE 121.

PROPERTIES OF NEGATIVE EXPONENTS.

306. We have required that negative exponents follow law A in order to obtain a meaning for them. It is possible to prove that negative exponents, having the meaning thus found, also follow laws B, C, D, and E in the operations of Algebra.

307. *To prove that negative exponents follow law B.*

Let n and r stand for two positive numbers, integral or fractional; then $-n$ and $-r$ are any two negative numbers.

CASE I. We will prove $a^n \div a^{-r} = a^{n-(-r)}$.

We know $a^n \div a^{-r} = a^n \times \frac{1}{a^{-r}}$,

by properties of fractions.

$= a^n \times a^{-(-r)}$,

by meaning of negative index.

$= a^{n-(-r)}$, by law *A*.

Therefore, $a^n \div a^{-r} = a^{n-(-r)}$. [13]

which is what was to be proved.

Of course $-(-r)$ can be written $+r$, and $n-(-r)$ can be written $n+r$. The form $n-(-r)$ is kept merely to show the subtraction of the negative index.

CASE II. We will prove $a^{-n} \div a^r = a^{-n-r}$.

We know $a^{-n} \div a^r = a^{-n} \times \frac{1}{a^r}$,

by properties of fractions.

$= a^{-n} \times a^{-r}$,

by meaning of negative index.

$= a^{-n-r}$, by law *A*.

Therefore, $a^{-n} \div a^r = a^{-n-r}$. [14]

which is what was to be proved.

CASE III. We will prove $a^{-n} \div a^{-r} = a^{-n-(-r)}$.

We know $a^{-n} \div a^{-r} = a^{-n} \times \frac{1}{a^{-r}}$,

by properties of fractions.

$= a^{-n} \times a^{-(-r)}$,

by meaning of negative index.

$= a^{-n-(-r)}$, by law *A*.

Therefore, $a^{-n} \div a^{-r} = a^{-n-(-r)}$, [15]

which is what was to be proved.

308. It should be noticed in the above demonstration of law *B* that *no restriction whatever is placed upon the relative magnitudes of the exponents n and r.* Consequently,

Law B is proved for all kinds of exponents, whether n is numerically greater than r or not.

Thus: $a^3 \div a^5 = a^{-2}$, $a^2 \div a^{10} = a^{-8}$, $a^{\frac{1}{2}} \div a^{\frac{3}{2}} = a^{-1}$, $a^{-2} \div a^3 = a^{-5}$, etc.

309. *To prove that negative exponents follow law C.*

Let n and r stand for any two positive numbers, integral or fractional.

CASE I. We will prove $(a^n)^{-r} = a^{-nr}$.

We know $(a^n)^{-r} = \frac{1}{(a^n)^r}$,

by meaning of negative index.

$= \frac{1}{a^{nr}}$,

by law *C* for positive indices.

$= a^{-nr}$,

by meaning of negative index.

Therefore, $(a^n)^{-r} = a^{-nr}$. [16]

which is what was to be proved.

CASE II. We will prove $(a^{-n})^r = a^{-nr}$.

We know $(a^{-n})^r = \left(\frac{1}{a^n}\right)^r$,

by meaning of negative index.

$= \frac{1}{a^{nr}}$,

by law *E* for positive indices.

$= a^{-nr}$,

by meaning of negative index.

Therefore, $(a^{-n})^r = a^{-nr}$. [17]

which is what was to be proved.

CASE III. We will prove $(a^{-n})^{-r}=a^{nr}$.

We know $(a^{-n})^{-r}=\frac{1}{(a^{-n})^{r}}$,

by meaning of negative index.

$=\frac{1}{a^{-nr}}$, by Case II.

$=a^{nr}$,

by meaning of negative index.

Therefore, $(a^{-n})^{-r}=a^{nr}$. [18]

which is what was to be proved.

310. *To prove that negative exponents follow law D.*

Let n be any positive number, integral or fractional.

We will prove $(abc)^{-n}=a^{-n}b^{-n}c^{-n}$.

We know $(abc)^{-n}=\frac{1}{(abc)^{n}}$,

by meaning of negative index.

$=\frac{1}{a^{n}b^{n}c^{n}}$,

by law D for positive indices.

$=\frac{1}{a^{n}}\,\frac{1}{b^{n}}\,\frac{1}{c^{n}}$,

by properties of fractions.

$=a^{-n}b^{-n}c^{-n}$,

by meaning of negative index.

Therefore, $(abc)^{-n}=a^{-n}b^{-n}c^{-n}$. [19]

which is what was to be proved.

311. *To prove that negative exponents follow law E.*

Let n be any positive number, integral or fractional.

We will prove $\left(\frac{a}{b}\right)^{-n}=\frac{a^{-n}}{b^{-n}}$.

We know $\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^{n}}$, by meaning of negative index.

$= \frac{1}{\frac{a^n}{b^n}}$, by law E for positive indices.

$= \frac{b^n}{a^n}$, by reducing fraction.

$= \frac{a^{-n}}{b^{-n}}$, by meaning of negative indices.

Therefore, $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}}$. [20]

which is what was to be proved.

EXERCISE 122.

EXAMPLES.

Perform the indicated operations in each of the following examples by means of the laws of exponents, now proved to hold for negative and fractional exponents.

1. $a^5 \times a^{-3}$.
2. $b^{-4} \times b^{-7}$.
3. $c^7 \times c^{-9}$.
4. $d^8 \times d^{-5}$.
5. $r^{12} \times r^{-10}$.
6. $n^{-11} \times n^8$.
7. $a^{-3}x \times a^{-5}y$.
8. $a^{-7}x \times a^5x^{-2}$.
9. $8a^{-4} \times 3a^2$.
10. $6ax^{-6} \times \frac{1}{2}bx^2$.
11. $3x^{-2}y^{\frac{1}{2}} \times 2x^2y^{\frac{1}{2}}$.
12. $a^{-\frac{1}{3}} \times a^{\frac{4}{3}}$.
13. $d^{-\frac{4}{5}} \times d^{-1}$.
14. $m^{-\frac{2}{3}} \times m^{-\frac{1}{3}}$.
15. $u^{-1} \times u^{\frac{3}{4}}$.
16. $6a^{-\frac{2}{r}}x \times 2a^{-\frac{1}{r}}x^{-1}$.
17. $(-7a^{-3}b^{-2})(-4a^2b^{-1})(-a^2b^2x^{-1})$.
18. $(2a^{\frac{1}{2}}b^{-\frac{2}{3}})(a^{-\frac{3}{2}}b^{\frac{2}{3}} - \frac{1}{2}a^{\frac{2}{3}}b^{\frac{3}{2}} + a^{\frac{3}{2}}b^{-\frac{2}{3}})$.
19. $a^8 \div a^{-3}$.
20. $b^{-4} \div b^{-9}$.
21. $c^{-8} \div c^{-5}$.
22. $a^{-5} \div a^{11}$.
23. $b^{-5} \div b^{-11}$.
24. $x^5 \div x^{-11}$.
25. $a^{-2}b^3 \div a^4b^{-5}$.
26. $ab^{-4} \div a^{-1}b^0$.
27. $a^{-3}b^{-n} \div a^{-4}b^{-r}$

23. $2a^0b^{-1}c^{-2} \div 3a^{-1}b^{-2}x^0$.

29. $3a^{-2}b^3c^0 \div 5a^5b^{-4}c^{-2}$.

30. $7a^{-1}b^{-2}c^{-3} \div 8a^{-2}b^{-3}c^{-4}$

31. $56x^5y^{-7}z^4 \div 7x^{-1}y^{-8}z^{-4}$.

32. $18a^{-\frac{1}{2}}b^{\frac{2}{3}}c^{-5} \div 6a^{\frac{3}{2}}b^{\frac{1}{3}}c^{-5}$.

33. $6x^{\frac{1}{3}}y^{-\frac{2}{3}}y^{\frac{1}{6}} \div 2x^{-\frac{2}{3}}y^{\frac{1}{3}}z^{-\frac{1}{2}}$.

34. $(a^{\frac{1}{2}}b + 3ab^2 + 5a^{\frac{3}{2}}b^3) \div (a^{\frac{1}{2}}b)$.

35. $(a^{-2})^{-3}$.

36. $(a^{-3})^2$.

37. $(a^{-3})^0$.

38. $(a^{-2})^{-5}$.

39. $(a^{-1})^{-1}$.

40. $(a^5)^{-2}$.

41. $(n^{-\frac{1}{2}})^4$.

42. $(n^{\frac{1}{3}})^{-3}$.

43. $(r^{-\frac{3}{4}})^{-\frac{2}{3}}$.

44. $(x^{\frac{2}{5}})^{-\frac{1}{2}}$.

45. $(c^{-8})^{\frac{3}{4}}$.

46. $(y^{-9})^{-\frac{2}{3}}$.

47. $(abc)^{-4}$.

48. $(a^2bc^{-3})^{-2}$.

49. $(a^{-4}b^2c^6)^{-3}$.

50. $(7a^{-6}x^{-9})^{-2}$.

51. $(x^{\frac{5}{6}}y^{\frac{3}{4}})^{-12}$.

52. $(8x^3y^{-6})^{-\frac{1}{3}}$.

53. $(a^{-2}b^3)^{\frac{1}{6}}$.

54. $(x^{-\frac{3}{4}}b^{\frac{5}{6}})^{\frac{10}{3}}$.

55. $(a^2d^{-\frac{1}{2}})^{-\frac{1}{2}}$.

56. $(a^{-5}b^{-10})^{-\frac{2}{5}}$.

57. $(-1x^3y^{-9})^{\frac{1}{3}}$.

58. $(-1x^3)^{-4}$.

59. $(-a^{-3})^4$.

60. $(-a^{-4})^{-3}$.

61. $(-a^6)^{-\frac{4}{3}}$.

62. $(-8a^{-3}b^{\frac{2}{4}}c^{-\frac{1}{3}})^{-\frac{1}{3}}$.

63. $\left(\frac{a}{b}\right)^{-5}$

64. $\left(\frac{a^2}{b^3}\right)^{-2}$

65. $\left(\frac{a^{-2}}{b^{-1}}\right)^3$

66. $\left(\frac{a^{-5}}{b^{-4}}\right)^{-7}$

67. $\left(\frac{a^4}{b^{-6}}\right)^{-2}$

68. $\left(\frac{a^4}{b^6}\right)^{-\frac{1}{2}}$

69. $\left(\frac{x^3}{y^9}\right)^{-\frac{2}{3}}$

70. $\left(\frac{x^{-8}}{y^4}\right)^{-\frac{3}{4}}$

71. $\left(\frac{a^{-2}b^3}{x^{-1}y^{-4}}\right)^2$

72. $\left(\frac{a^{-3}b}{x^3y^{-2}}\right)^{-3}$

73. $\left(\frac{6a^{-4}b^0}{5x^2y^{-2}}\right)^{-2}$

74. $\left(\frac{a^0b^{-2}}{xy^3}\right)^3$

75. $\left(\frac{x+y}{x-y}\right)^{-2}$

76. $\left(\frac{a^5b^{-3}}{3x^{-1}y^2}\right)^{-1}$

77. $\left(\frac{a^6b^{-3}}{x^{-9}y^{\frac{3}{4}}}\right)^{-\frac{1}{3}}$

78. $(3x^{\frac{2}{3}}-\frac{3}{2}x^{\frac{1}{3}}+4)\times 2x^{-\frac{1}{3}}$.

79. $(ax^{-4}+bx^{-3}+cx^{-2}+dx^{-1}+ex^{0})\times x^{4}$.

80. $(a^{2}x^{-1}+3a^{3}x^{-2})(4a^{-1}-5x^{-1}+6ax^{-2})$.

$$
\begin{array}{l}
4a^{-1}-5x^{-1}+6ax^{-2} \\
a^{2}x^{-1}+3a^{3}x^{-2} \\
\hline
4ax^{-1}-5a^{2}x^{-2}+\ 6a^{3}x^{-3} \\
\qquad 12a^{2}x^{-2}-15a^{3}x^{-3}+18a^{4}x^{-4} \\
\hline
4ax^{-1}+7a^{2}x^{-2}-\ 9a^{3}x^{-3}+18a^{4}x^{-4}
\end{array}
$$

81. $(2x^{-\frac{1}{5}}-3x^{0}+4x^{\frac{1}{5}})(x^{-\frac{3}{5}}-2x^{-\frac{2}{5}}+3x^{-\frac{1}{5}})$.

$$
\begin{array}{l}
x^{-\frac{3}{5}}-2x^{-\frac{2}{5}}+\ 3x^{-\frac{1}{5}} \\
2x^{-\frac{1}{5}}-3x^{0}\ +\ 4x^{\frac{1}{5}} \\
\hline
2x^{-\frac{4}{5}}-4x^{-\frac{3}{5}}+\ 6x^{-\frac{2}{5}} \\
\qquad -3x^{-\frac{3}{5}}+\ 6x^{-\frac{2}{5}}-\ 9x^{-\frac{1}{5}} \\
\qquad\qquad\quad 4x^{-\frac{2}{5}}-\ 8x^{-\frac{1}{5}}+12x^{0} \\
\hline
2x^{-\frac{4}{5}}-7x^{-\frac{3}{5}}+16x^{-\frac{2}{5}}-17x^{-\frac{1}{5}}+12
\end{array}
$$

82. $(x^{-\frac{1}{2}}+y^{-2})(x^{-\frac{1}{2}}-y^{-2})$.

83. $(x^{\frac{1}{4}}y+y^{\frac{2}{3}})(x^{\frac{1}{4}}-y^{-\frac{1}{3}})$.

84. $(x+x^{\frac{1}{3}}+x^{-\frac{1}{3}})(x^{\frac{1}{3}}+x^{-\frac{1}{3}}-x^{-1})$.

85. $(x^{-\frac{2}{3}}+x^{-\frac{1}{3}}+1)(x^{-\frac{1}{3}}-1)$.

86. $(x^{-3}+x^{-\frac{3}{2}}+1)(x^{-3}-x^{-\frac{3}{2}}-1)$.

87. $(x^{-\frac{4}{3}}-2x^{-\frac{2}{3}}y^{\frac{2}{3}}+y^{\frac{4}{3}})(x^{-\frac{2}{3}}-y^{\frac{2}{3}})$.

88. $(x^{-\frac{3}{2}}-x^{-1}+x^{-\frac{1}{2}}-1)(x^{-\frac{1}{2}}+1)$.

89. $(2a^{\frac{3}{2}}-3ax^{\frac{1}{2}})(3a^{-\frac{1}{2}}+2x^{-\frac{1}{2}})(4a^{\frac{1}{2}}x^{\frac{1}{2}}+9a^{-\frac{1}{2}}x^{\frac{3}{2}})$.

90. $(a^{\frac{2}{3}}+b^{\frac{4}{3}}+c^{2}-cb^{\frac{2}{3}}-ca^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{2}{3}})(a^{\frac{1}{3}}+b^{\frac{2}{3}}+c)$.

91. $(x^{-\frac{3}{2}}-x^{-1}y^{\frac{1}{2}}+x^{-\frac{1}{2}}y-y^{\frac{3}{2}})\div(x^{-\frac{1}{2}}-y^{\frac{1}{2}})$.

$$
\begin{array}{l}
x^{-\frac{1}{2}}-y^{\frac{1}{2}}\ )\ x^{-\frac{3}{2}}-x^{-1}y^{\frac{1}{2}}+x^{-\frac{1}{2}}y-y^{\frac{3}{2}}\ (\ x^{-1}+y \\
\qquad\quad x^{-\frac{3}{2}}-x^{-1}y^{\frac{1}{2}} \\
\qquad\quad\ \overline{\qquad\qquad\qquad} \\
\qquad\qquad\qquad\quad x^{-\frac{1}{2}}y-y^{\frac{3}{2}} \\
\qquad\qquad\qquad\quad x^{-\frac{1}{2}}y-y^{\frac{3}{2}} \\
\qquad\qquad\qquad\quad\ \overline{\qquad\qquad}
\end{array}
$$

92. $(64x^{-1}+27y^{-2})\div(4x^{-\frac{1}{3}}+3y^{-\frac{2}{3}})$.

93. $(x^{-1}-y^{-1})\div(x^{-\frac{1}{3}}-y^{-\frac{1}{3}})$.

94. $(x^{-3}+2x^{-2}-3x^{-1})\div(x^{-2}+3x^{-1})$.

95. $(x^4+x^{-4}+4[x^2+x^{-2}]+6)\div(x^2+x^{-2}+2)$.

Arrange terms according to powers of x, so that the exponents of dividend will be in the *descending* order, 4, 2, 0, −2, −4.

96. $(x^{-\frac{5}{2}}-x^{-2}-4x^{-\frac{3}{2}}+6x^{-1}-2x^{-\frac{1}{2}})\div(x^{-\frac{3}{2}}-4x^{-\frac{1}{2}}+2)$

97. $(x^{\frac{4}{3}}+2x^{\frac{2}{3}}+1-x^{-\frac{4}{3}})\div(x^{\frac{1}{3}}+x^{-\frac{1}{3}}-x^{-1})$.

98. $(x^{\frac{3n}{2}}-x^{-\frac{3n}{2}})\div(x^{\frac{n}{2}}-x^{-\frac{n}{2}})$.

99. Simplify $\dfrac{(x^{2n})^{-\frac{1}{n}}}{(x^{2n})^{-\frac{1}{r}}}$.

100. Simplify $\dfrac{n(rx)^{\frac{1}{2}}(nbx)^{0}}{(ax)^{\frac{1}{2}}\sqrt{nra}}$.

101. Simplify $[(a^{\frac{4}{3}}b^{\frac{3}{4}})^{-\frac{1}{2}}\times(a^{-\frac{3}{4}}b^{-\frac{4}{3}})^{\frac{1}{2}}]^{-24}$.

102. Simplify $[ab^2(ab^3)^{\frac{1}{2}}(a^2b^3)^{\frac{1}{6}}]^{\frac{1}{2}}$.

103. Simplify $(2^{\frac{1}{4}}a+3^{\frac{1}{4}}b)(3^{\frac{1}{4}}a-2^{\frac{1}{4}}b)-6^{\frac{1}{4}}(a^2-b^2)+2^{\frac{1}{2}}ab$.

CHAPTER XVII.

SURDS.

EXERCISE 123.

Definitions and General Principles.

312. From the last chapter the student has learned that there are two methods in use for indicating the root of an expression, one by the ordinary radical sign and the other by a fractional exponent. While it is unnecessary to have two ways of writing the same thing, yet each method of notation has special advantages in particular cases, which accounts for the use of the two methods. Of course the same laws (namely, A, B, C, D, and E of the last chapter) govern the operations with roots, whatever form of notation be used.

313. When a root of an arithmetical numeral can only be found approximately, that root is called a **Surd.** Thus, $\sqrt{2}$ and $\sqrt[3]{5}$ are surds. Expressions like $\sqrt{4}$, $\sqrt[3]{8}$, etc., are said to be in the *form of a surd.* Expressions like $\sqrt{a}$, $\sqrt[3]{ab}$, etc., are often called surds, although, of course, they are only such when the letters stand for numbers whose roots cannot be exactly taken.

314. Surds are of the same **Order** when the same root is required to be taken in each. Thus, $\sqrt{2}$ and $\sqrt{3}$ are of the **Second Order,** $\sqrt[3]{3}$ and $\sqrt[3]{2}$ are of the **Third Order**, etc. Surds of the second and third orders are often called *Quadratic* and *Cubic Surds* respectively.

315. Monomials, binomials, trinomials, etc., which contain surds are called respectively **Monomial Surds, Binomial Surds, Trinomial Surds,** etc. Thus, $3\sqrt[3]{5}$ is a monomial surd, $2+\sqrt{5}$ is a binomial surd, and $\sqrt{3}-\sqrt{2}+\sqrt{5}$ is a trinomial surd.

316. A factor written before a surd is often called the **Coefficient of the Surd.** Thus, in $2a\sqrt{5}$, $2a$ is called the coefficient of the surd $\sqrt{5}$.

317. Any expression containing surds is called an **Irrational Expression,** and any expression not containing surds is called a **Rational Expression.**

318. The operations with surds depend upon principles established in the last chapter. For convenience of reference, we will restate below those principles which we shall make use of in the present chapter.

319. *The r th root of the product of several numbers is equal to the product of the r th roots of the several numbers.*

That is, $$\sqrt[r]{abc}=\sqrt[r]{a}\sqrt[r]{b}\sqrt[r]{c}, \qquad [1]$$

because $$(abc)^{\frac{1}{r}}=a^{\frac{1}{r}}b^{\frac{1}{r}}c^{\frac{1}{r}},$$

by equation [10], Chapter XVI.

320. *The r th root of the quotient of two numbers is equal to the quotient of their r th roots.*

That is, $$\sqrt[r]{\frac{a}{b}}=\frac{\sqrt[r]{a}}{\sqrt[r]{b}} \qquad [2]$$

because $$\left(\frac{a}{b}\right)^{\frac{1}{r}}=\frac{a^{\frac{1}{r}}}{b^{\frac{1}{r}}},$$

by equation [11], Chapter XVI.

321. *The rtth root of a number equals the rth root of the tth root of the number.*

That is, $\sqrt[rt]{a} = \sqrt[r]{\sqrt[t]{a}}$, [3]

because $a^{\frac{1}{rt}} = (a^{\frac{1}{t}})^{\frac{1}{r}}$,

by equation [9], Chapter XVI.

322. *The rtth root of the ntth power of a number equals the rth root of the nth power of that number.*

That is, $\sqrt[rt]{a^{nt}} = \sqrt[r]{a^{n}}$ [4]

because $a^{rt} = a^{r}$,

by equation [5], Chapter XVI.

323. *The nth power of the rth root of a number equals the rth root of the nth power of that number.*

That is, $(\sqrt[r]{a})^n = \sqrt[r]{a^n}$. [5]

This is equation [4], Chapter XVI.

EXERCISE 124.

To Remove a Factor from Beneath the Radical Sign.

324. When any factor of the number under the radical sign is an exact power of the indicated root, the root of that factor may be extracted and written as the coefficient of the surd, while the other factors are left under the radical sign.

(1) Thus, $\sqrt{8} = \sqrt{4 \times 2}$
$= \sqrt{4}\sqrt{2}$ by [1].
$= 2\sqrt{2}$

(2) Also, $\sqrt[3]{81} = \sqrt[3]{27 \times 3}$
$= \sqrt[3]{27}\sqrt[3]{3}$ by [1].
$= 3\sqrt[3]{3}$

(3) Also, $\sqrt[3]{16ax^4} = \sqrt[3]{8x^3 \times 2ax}$
$= \sqrt[3]{8x^3}\sqrt[3]{2ax}$ by [1].
$= 2x\sqrt[3]{2ax}$

EXAMPLES.

Remove as many factors as possible from beneath the radical sign in each of the following:

1. $\sqrt{12}$. 9. $\sqrt[3]{56}$. 17. $5\sqrt[4]{144}$. 25. $\sqrt[8]{a^{12}b^{15}}$.
2. $\sqrt{28}$. 10. $\sqrt[5]{160}$. 18. $\sqrt{a^2b}$. 26. $\sqrt{4a^2b}$.
3. $\sqrt{50}$. 11. $\sqrt[5]{2048}$. 19. $\sqrt[3]{m^3x^2}$. 27. $\sqrt{25m^4x}$.
4. $\sqrt{72}$. 12. $\sqrt[6]{3645}$. 20. $\sqrt[4]{b^8c^3}$. 28. $\sqrt{36a^3b^2}$.
5. $\sqrt[3]{72}$. 13. $\sqrt[4]{567}$. 21. $\sqrt[4]{a^4c}$. 29. $\sqrt[3]{64x^6y}$.
6. $\sqrt[3]{500}$. 14. $\sqrt[4]{112}$. 22. $\sqrt[5]{c^{15}x^2}$. 30. $\sqrt[4]{81m^6x^2}$.
7. $\sqrt[3]{108}$. 15. $2\sqrt{405}$. 23. $\sqrt[3]{x^6y^4}$. 31. $\sqrt[6]{64a^8b^6}$.
8. $\sqrt[3]{192}$. 16. $3\sqrt[3]{864}$. 24. $\sqrt{y^4z^3}$. 32. $2\sqrt{80c^4d}$.

EXERCISE 125.

TO INTRODUCE THE COEFFICIENT OF A SURD UNDER THE RADICAL SIGN.

325. It is sometimes convenient to have a surd in a form without a coefficient. The coefficient can always be introduced under the radical sign by reversing the process of Art. 324.

(1) Thus, $2\sqrt{6} = \sqrt{2^2}\sqrt{6}$
$= \sqrt{2^2 \times 6}$ by [1].
$= \sqrt{24}$

(2) Also, $50\sqrt{50} = \sqrt{50^2}\sqrt{50}$
$= \sqrt{50^2 \times 50}$ by [1].
$= \sqrt{125000}$

(3) Also, $4\sqrt[3]{5} = \sqrt[3]{4^3}\sqrt[3]{5}$
$= \sqrt[3]{4^3 \times 5}$ by [1].
$= \sqrt[3]{320}$

326. As the same process may evidently be applied in any case, we say:

Any coefficient of a surd may be introduced as a factor under the radical sign, provided the coefficient be first raised to a power equal to the index of the surd.

EXAMPLES.

Place the coefficient of the surd beneath the radical sign in each of the following:

1. $2\sqrt{2}$. 8. $4\sqrt[4]{4}$. 15. $x\sqrt{2}$. 22. $-2\sqrt[5]{x}$.
2. $5\sqrt{7}$. 9. $2\sqrt[5]{3}$. 16. $y\sqrt[5]{6}$. 23. $a\sqrt{b}$.
3. $6\sqrt{5}$. 10. $3\sqrt[7]{3}$. 17. $z\sqrt[3]{7}$. 24. $m^2\sqrt{yz^2}$.
4. $2\sqrt{21}$. 11. $10\sqrt[3]{4}$. 18. $(-b)\sqrt{6}$ 25. $\frac{1}{2}\sqrt{3}$.
5. $3\sqrt[3]{2}$. 12. $9\sqrt[3]{10}$. 19. $(-d)\sqrt[5]{7}$ 26. $\frac{2}{3}\sqrt[3]{m^2}$.
6. $7\sqrt[3]{3}$. 13. $8\sqrt{41}$. 20. $-n\sqrt[5]{10}$ 27. $\frac{3}{7}\sqrt[3]{7e^2}$.
7. $8\sqrt[3]{5}$. 14. $a\sqrt{5}$. 21. $-5\sqrt{a}$. 28. $\frac{4}{9}\sqrt{18y^5}$.

EXERCISE 126.

TO INTEGRALIZE THE EXPRESSION UNDER A RADICAL SIGN.

327. The expression under the radical sign of any surd can always be made a whole number.

(1) Thus,
$$\sqrt{\tfrac{1}{2}}=\sqrt{\tfrac{1}{2}\times\tfrac{2}{2}}=\sqrt{\tfrac{2}{4}}$$
$$=\sqrt{\tfrac{1}{4}\times 2}$$
$$=\sqrt{\tfrac{1}{4}}\sqrt{2} \qquad \text{by [1].}$$
$$=\tfrac{1}{2}\sqrt{2} \qquad \text{because } \sqrt{\tfrac{1}{4}}=\tfrac{1}{2}.$$

(2) Also,
$$\sqrt[3]{\tfrac{2}{3}}=\sqrt[3]{\tfrac{2}{3}\times\tfrac{9}{9}}=\sqrt[3]{\tfrac{18}{27}}$$
$$=\sqrt[3]{\tfrac{1}{27}\times 18}$$
$$=\sqrt[3]{\tfrac{1}{27}}\sqrt[3]{18} \qquad \text{by [1].}$$
$$=\tfrac{1}{3}\sqrt[3]{18}$$

(3) Also,
$$\sqrt[4]{\tfrac{7}{8}} = \sqrt[4]{\tfrac{7}{8}\times\tfrac{2}{2}} = \sqrt[4]{\tfrac{14}{16}}$$
$$= \sqrt[4]{\tfrac{1}{16}\times 14}$$
$$= \sqrt[4]{\tfrac{1}{16}}\sqrt[4]{14} \qquad \text{by [1].}$$
$$= \tfrac{1}{2}\sqrt[4]{14}$$

(4) Also,
$$\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{a}{b}\times\frac{b^{n-1}}{b^{n-1}}} = \sqrt[n]{\frac{ab^{n-1}}{b^n}}$$
$$= \sqrt[n]{\frac{1}{b^n}\times ab^{n-1}}$$
$$= \sqrt[n]{\frac{1}{b^n}}\sqrt[n]{ab^{n-1}} \qquad \text{by [1].}$$
$$= \frac{1}{b}\sqrt[n]{ab^{n-1}}$$

328. As the same process can evidently be applied in any case, we may say:

The expression under the radical sign in any surd can be made integral by multiplying both numerator and denominator by such a number as will render the denominator a perfect power of the indicated root, and then taking the root of the denominator thus found.

EXAMPLES.

Integralize the expression under the radical sign in each of the following:

1. $\sqrt{\frac{3}{4}}$.
2. $\sqrt{\frac{1}{3}}$.
3. $\sqrt{\frac{3}{4}}$.
4. $\sqrt{\frac{7}{8}}$.
5. $\sqrt{\frac{3}{7}}$.
6. $\sqrt{\frac{9}{8}}$.
7. $\sqrt{\frac{5}{14}}$.
8. $\sqrt{\frac{1}{207}}$.
9. $\sqrt[3]{\frac{2}{3}}$.
10. $\sqrt[3]{\frac{3}{4}}$.
11. $\sqrt[4]{\frac{1}{8}}$.
12. $\sqrt[4]{\frac{5}{3}}$.
13. $\sqrt[4]{\frac{7}{8}}$.
14. $\sqrt[3]{\frac{24}{25}}$.
15. $\sqrt{\frac{1088}{3}}$.
16. $15\sqrt{\frac{27}{50}}$.
17. $\frac{3}{2}\sqrt[3]{\frac{30}{20}}$.
18. $35\sqrt{\frac{2}{147}}$.
19. $\sqrt{22\frac{2}{3}}$.
20. $\frac{2}{3}\sqrt{\frac{1}{8}}$.
21. $\sqrt{\frac{a}{b}}$.
22. $\sqrt[3]{\frac{x}{y}}$.
23. $\sqrt{\frac{m^3}{49}}$.
24. $\frac{1}{2}\sqrt[5]{\frac{y^4}{x^3}}$.
25. $\frac{1}{a}\sqrt[3]{\frac{n}{r^2}}$.
26. $\sqrt{\frac{ab^8}{2x^3}}$.
27. $\sqrt[3]{\frac{ab^2}{xy^2}}$.
28. $\frac{2a}{3}\sqrt[3]{\frac{3x}{4a^2y}}$.

EXERCISE 127.

TO LOWER OR RAISE THE INDEX OF A SURD.

329. We can change the index of any surd in the following manner:

(1) Thus, $\sqrt[4]{4} = \sqrt{\sqrt{4}}$ by [3].

$= \sqrt{2}$ since $\sqrt{4} = 2$.

(2) Also, $\sqrt[6]{1000} = \sqrt{\sqrt[3]{1000}}$ by [3].

$= \sqrt{10}$ since $\sqrt[3]{1000} = 10$.

(3) Also, $\sqrt[6]{256c^2x^8} = \sqrt[3]{\sqrt{256c^2x^8}}$ by [3].

$= \sqrt[3]{16cx^4}$, since $\sqrt{256c^2x^8} = 16cx^4$

330. Since equation [3] is true in all cases, we know *The index of a surd can be lowered if the expression under the radical sign is a perfect power corresponding to some factor of the original radical index.*

EXAMPLES.

Lower the index of each of the following surds:

1. $\sqrt[4]{36}$.
2. $\sqrt[6]{2500}$.
3. $\sqrt[4]{16}$.
4. $\sqrt[8]{16}$.
5. $\sqrt[8]{1296}$
6. $\sqrt[9]{1000}$.
7. $\sqrt[10]{256^2}$.
8. $\sqrt[10]{288^2}$.
9. $\sqrt[9]{27}$.
10. $\sqrt[6]{27a^3}$.
11. $\sqrt[4]{a^2b^6}$.
12. $\sqrt[24]{256}$.
13. $\sqrt[21]{128}$.
14. $\sqrt[6]{8000}$.
15. $\sqrt[4]{6\frac{1}{4}}$.
16. $\sqrt[6]{343x^3y^9}$.
17. $\sqrt[4]{\frac{4}{9}}$.
18. $\sqrt[6]{\frac{216}{343}}$.
19. $3\sqrt[4]{49a^2b^6c^8}$
20. $4\sqrt[6]{\frac{81}{144}a^2x^4}$.

EXERCISE 128.

TO REDUCE A SURD TO ITS SIMPLEST FORM.

331. A surd is in its **Simplest Form** when, (1) no factor of the expression under the radical sign is a perfect power of the required root; (2) the expression under the radical sign is integral; (3) the index of the surd is the lowest possible.

332. Methods of making the different reductions required by this definition have already been explained in exercises 124, 126, and 127. We give a few examples.

(1) Simplify $\sqrt[9]{\frac{a^3}{8b^6}}$.

$$\sqrt[9]{\frac{a^3}{8b^6}} = \sqrt[3]{\frac{a}{2b^2}} \quad \text{by exercise 127.}$$

$$= \frac{1}{2b}\sqrt[3]{4ab} \quad \text{by exercise 126.}$$

(2) Simplify $\sqrt[4]{\frac{4\,0\,0}{9}}$.

$$\sqrt[4]{\frac{4\,0\,0}{9}} = \sqrt{\frac{2\,0}{3}} \quad \text{by exercise 127.}$$

$$= \tfrac{1}{3}\sqrt{60} \quad \text{by exercise 126.}$$

$$= \tfrac{2}{3}\sqrt{15} \quad \text{by exercise 124.}$$

(3) Simplify $\frac{5}{2}\sqrt[6]{\frac{5}{1}\frac{1}{2}\frac{2}{5}}$.

$$\tfrac{5}{2}\sqrt[6]{\tfrac{5}{1}\tfrac{1}{2}\tfrac{2}{5}} = \tfrac{5}{2}\sqrt{\tfrac{8}{5}} \quad \text{by exercise 127.}$$

$$= 5\sqrt{\tfrac{2}{5}} \quad \text{by exercise 124.}$$

$$= \sqrt{10} \quad \text{by exercise 126.}$$

333. In any piece of work it is usually expected that all the surds will finally be left in their simplest form.

EXAMPLES.

Reduce each of the following surds to its simplest form:

1. $\sqrt{\frac{9}{5}}$. 3. $\sqrt[4]{\frac{2\,5}{4}}$. 5. $\sqrt[6]{\frac{6\,4}{8\,1}}$. 7. $\sqrt[4]{\frac{a^2}{c^4x^6}}$.

2. $\sqrt[3]{\frac{1\,6}{3}}$. 4. $\sqrt[6]{\frac{6\,4}{2\,7}}$. 6. $\sqrt[3]{\frac{5}{1\,2}}$. 8. $\sqrt{1-\frac{1}{x^2}}$.

EXERCISE 129.

ADDITION AND SUBTRACTION OF SURDS.

334. Surds which differ only in their coefficients are said to be **Similar.** Thus, $6\sqrt{2}$ and $15\sqrt{2}$ are similar surds; also $\frac{2}{3}\sqrt[3]{\frac{5}{7}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{7}}$; also $5\sqrt[5]{ab^2}$ and $n\sqrt[5]{ab^2}$.

335. The addition and subtraction of surds involves no principle not already used in the addition and subtraction of other expressions, as the following examples show:

(1) Combine the terms of $10\sqrt{7}-3\sqrt{7}+5\sqrt{7}$.

$$10\sqrt{7}-3\sqrt{7}+5\sqrt{7}=12\sqrt{7},$$

by the usual process of addition of terms.

(2) Combine the terms of $7\sqrt{2}-\sqrt{18}+2\sqrt{8}$.

Putting each surd in its simplest form, we have

$$7\sqrt{2}-\sqrt{18}+2\sqrt{8}=7\sqrt{2}-3\sqrt{2}+4\sqrt{2}$$
$$=8\sqrt{2}$$

by the usual process of addition of terms.

(3) Combine the terms of $5\sqrt[3]{4}+2\sqrt[3]{32}-\sqrt[3]{108}$.

Putting each surd in its simplest form, we have

$$5\sqrt[3]{4}+2\sqrt[3]{32}-\sqrt[3]{108}=5\sqrt[3]{4}+4\sqrt[3]{4}-3\sqrt[3]{4}$$
$$=6\sqrt[3]{4}$$

(4) Combine the terms of $\frac{3}{2}\sqrt{\frac{2}{3}}-\frac{2}{3}\sqrt{\frac{3}{2}}$.

Putting each surd in its simplest form, we have

$$\tfrac{3}{2}\sqrt{\tfrac{2}{3}}-\tfrac{2}{3}\sqrt{\tfrac{3}{2}}=\tfrac{1}{2}\sqrt{6}-\tfrac{1}{3}\sqrt{6}$$
$$=\tfrac{1}{6}\sqrt{6}$$

(5) Combine terms of $22\sqrt[3]{a^6b}-3a^2\sqrt[3]{64b}+5a\sqrt[3]{a^3b}$.

Putting each surd in its simplest form, we have

$$22\sqrt[3]{a^6b}-3a^2\sqrt[3]{64b}+5a\sqrt[3]{a^3b}$$
$$=22a^2\sqrt[3]{b}-12a^2\sqrt[3]{b}+5a^2\sqrt[3]{b}$$
$$=15a^2\sqrt[3]{b}$$

336. We observe the advantage of reducing each of the surds in any given expression to its simplest form, for then it can be told whether or not some of the surds are similar to each other, and consequently whether or not they can be combined together; for *only similar surds can be combined into a single surd.*

EXAMPLES.

Express each of the following expressions in as few terms as possible :

1. $2\sqrt{3}+3\sqrt{3}$.
2. $7\sqrt{2}-3\sqrt{2}$.
3. $11\sqrt{13}-4\sqrt{13}$.
4. $7\sqrt{2}+\sqrt{18}$.
5. $4\sqrt[3]{4}-3\sqrt[3]{4}+2\sqrt[3]{4}$.
6. $2\sqrt[3]{32}-\sqrt[3]{108}$.
7. $4\sqrt[5]{2}-\sqrt[5]{64}$.
8. $\sqrt[4]{243}-5\sqrt[4]{48}$.
9. $\sqrt{\frac{45}{4}}-\sqrt{20}$.
10. $\frac{1}{6}\sqrt{245}+5\sqrt{\frac{1}{5}}$.
11. $\sqrt[3]{\frac{4}{27}}-\sqrt[3]{\frac{10}{2}}-3\sqrt[3]{256}+\sqrt[3]{625}$.
12. $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{4}{15}}$.
13. $2+\frac{3}{2}\sqrt{\frac{2}{3}}-\frac{2}{3}\sqrt{\frac{3}{2}}$.
14. $\frac{2}{5}\sqrt{\frac{3}{5}}+\frac{7}{8}\sqrt{\frac{20}{147}}-\frac{1}{2}\sqrt[4]{\frac{25}{9}}$.
15. $\sqrt{a^3}+2\sqrt{a}-\sqrt{4a}$.
16. $\sqrt[3]{a^2}+\frac{1}{2}\sqrt[3]{a^2}-3\sqrt[3]{27a^2}$.
17. $\sqrt[3]{27c^4}-\sqrt[3]{8c^4}+\sqrt[3]{125c}$.
18. $\sqrt{a^4x}+\sqrt{b^4x}-\sqrt{4a^2b^2x}$.
19. $6a\sqrt{63ab^3}-3\sqrt{112a^3b^3}+5b\sqrt{28a^3b}$.
20. $b\sqrt{\frac{4a}{b^4}}-\sqrt{\frac{9a}{b^2}}+\frac{1}{b}\sqrt{\frac{a}{4}}+2b\sqrt{\frac{25a}{4b^4}}$.

EXERCISE 130.

MULTIPLICATION AND DIVISION OF SURDS.

337. The product of any number of surds of the *same index* can always be expressed as a single surd by means of equation [1].

(1) Find the product of $\sqrt{2}\times\sqrt{5}\times\sqrt{7}$.

$$\sqrt{2}\times\sqrt{5}\times\sqrt{7}=\sqrt{2\times5\times7} \qquad \text{by [1].}$$
$$=\sqrt{70}$$

(2) Find the product of $\sqrt{2} \times \sqrt{18}$.

$$\sqrt{2} \times \sqrt{18} = \sqrt{2 \times 18} \qquad \text{by [1].}$$
$$= 6$$

(3) Find the product of $\sqrt[5]{54} \times \sqrt[5]{9}$.

$$\sqrt[5]{54} \times \sqrt[5]{9} = \sqrt[5]{54 \times 9} \qquad \text{by [1].}$$
$$= 3\sqrt[5]{2}$$

The result should always appear in its *simplest form*.

338. The quotient of two surds of the *same index* may be expressed by means of equation [2].

(1) Find the quotient of $\sqrt{28} \div \sqrt{7}$.

$$\sqrt{28} \div \sqrt{7} = \sqrt{\tfrac{28}{7}} \qquad \text{by [2].}$$
$$= \sqrt{4} = 2$$

(2) Find the quotient of $\sqrt[3]{81} \div \sqrt[3]{6}$.

$$\sqrt[3]{81} \div \sqrt[3]{6} = \sqrt[3]{\tfrac{81}{6}} \div \sqrt[3]{\tfrac{27}{2}} \qquad \text{by [2]}$$
$$= 3\sqrt[3]{\tfrac{1}{2}} \qquad \text{by exercise 124.}$$
$$= \tfrac{3}{2}\sqrt[3]{4} \qquad \text{by exercise 126.}$$

The result should always appear in its *simplest form*.

339. If the product or quotient of surds of *different indices* is sought, the surds may first be reduced to a common index by exercise 127.

(1) Find the product of $\sqrt{5} \times \sqrt[3]{4}$.

$$\sqrt{5} \times \sqrt[3]{4} = \sqrt[6]{125} \times \sqrt[6]{16} \qquad \text{by exercise 127.}$$
$$= \sqrt[6]{125 \times 16} = \sqrt[6]{2000} \qquad \text{by [1].}$$

(2) Find the quotient of $\sqrt[3]{9} \div \sqrt{3}$.

$$\sqrt[3]{9} \div \sqrt{3} = \sqrt[6]{81} \div \sqrt[6]{27} \qquad \text{by exercise 127.}$$
$$= \sqrt[6]{\tfrac{81}{27}} = \sqrt[6]{3} \qquad \text{by [2].}$$

(3) Find the product of $\sqrt{ab^3} \times \sqrt[5]{a^2b}$.

$$\sqrt{ab^3} \times \sqrt[5]{a^2b} = \sqrt[10]{a^5b^{15}} \times \sqrt[10]{a^4b^2} \quad \text{by Ex. 127.}$$
$$= \sqrt[10]{a^9b^{17}} \quad \text{by [1].}$$
$$= b\sqrt[10]{a^9b^7} \quad \text{by exercise 124.}$$

(4) Find the quotient of $\sqrt[5]{\frac{4}{5}} \times 2\sqrt[10]{\frac{1}{400}}$.

$$\sqrt[5]{\tfrac{4}{5}} \div 2\sqrt[10]{\tfrac{1}{400}} = \sqrt[10]{\tfrac{16}{25}} \div 2\sqrt[10]{\tfrac{1}{400}} \quad \text{by Ex. 127.}$$
$$= \tfrac{1}{2}\sqrt[10]{\tfrac{16}{25} \times \tfrac{400}{1}} \quad \text{by [2].}$$
$$= \tfrac{1}{2}\sqrt[10]{16 \times 16} = \tfrac{1}{2}\sqrt[5]{16} \quad \text{by Ex. 127.}$$

EXAMPLES.

Perform the indicated operations in each of the following and leave the results in their simplest form:

1. $\sqrt{5} \times \sqrt{20}$.
2. $\sqrt[3]{3} \times \sqrt[3]{18}$.
3. $\sqrt[3]{6} \times \sqrt[3]{32}$.
4. $\sqrt{2} \times \sqrt{12}$.
5. $\sqrt[5]{4} \times \sqrt[5]{8}$.
6. $\sqrt[4]{6} \times \sqrt[4]{8}$.
7. $\frac{1}{2}\sqrt[3]{4} \times \sqrt[3]{10}$.
8. $\sqrt[3]{81} \times \sqrt[3]{45}$.
9. $\sqrt[3]{7} \times \sqrt[3]{-49}$.
10. $\sqrt[3]{50} \times \sqrt[3]{40}$.
11. $\sqrt[6]{11} \times \sqrt[6]{121}$.
12. $2\sqrt{a} \times \sqrt{9a^3}$.
13. $a\sqrt{y} \times b\sqrt{y^3}$.
14. $\sqrt{15c^3} \times \sqrt{5c}$.
15. $2\sqrt[3]{b^5} \times 3\sqrt[3]{b^4}$.
16. $\sqrt{48} \div \sqrt{3}$.
17. $\sqrt{45} \div \sqrt{10}$.
18. $\sqrt{50} \div \sqrt{6}$.
19. $\sqrt{96} \div \frac{1}{3}\sqrt{3}$.
20. $\sqrt[3]{81} \div \sqrt[3]{6}$.
21. $\sqrt[4]{80} \div \sqrt[4]{20}$.
22. $\sqrt{8b^9} \div \sqrt{2b^5}$.
23. $\sqrt[3]{54a^7} \div \sqrt[3]{2a}$.
24. $\sqrt{\frac{5}{6}} \div \sqrt{\frac{3}{5}}$.
25. $\sqrt{\frac{12}{35}} \div \sqrt{\frac{21}{5}}$.
26. $\sqrt{\frac{2}{9}} \div \sqrt{\frac{3}{2}}$.
27. $\sqrt{\frac{22}{27}} \div \sqrt{\frac{6}{11}}$.
28. $\sqrt{5} \times \sqrt[3]{4}$.
29. $\sqrt{125} \times \sqrt[4]{36}$.
30. $\sqrt{\frac{2}{3}} \times \sqrt[3]{\frac{3}{2}}$.
31. $\sqrt{\frac{1}{2}} \times \sqrt[3]{\frac{1}{4}}$.
32. $\sqrt[4]{64} \times \sqrt[8]{16}$.
33. $\sqrt[5]{12} \times \sqrt[10]{96}$.
34. $\sqrt[5]{\frac{3}{8}} \times \sqrt[3]{\frac{2}{3}}$.
35. $\sqrt[4]{\frac{7}{9}} \times \sqrt[5]{\frac{15}{7}}$.
36. $b\sqrt{a} \times \sqrt[4]{a^2b^3}$.
37. $a\sqrt[3]{4a} \times \sqrt[3]{2a^3}$.
38. $\sqrt[3]{b^4c^2} \times \sqrt[5]{b^4c^4}$
39. $3\sqrt{2c} \times 5\sqrt[4]{2c^2}$.
40. $\sqrt[3]{9} \div \sqrt{3}$.
41. $\sqrt[3]{72} \div \sqrt{6}$.
42. $\sqrt[3]{12} \div \sqrt{6}$.
43. $\sqrt{\frac{1}{2}} \div \sqrt[3]{\frac{1}{2}}$.
44. $\sqrt[3]{24} \div 6\sqrt[6]{3}$.
45. $\sqrt{\frac{8}{45}} \div \sqrt[4]{6\frac{1}{4}}$.
46. $5 \div \sqrt{3}$.
47. $30 \div \sqrt{210}$.
48. $\sqrt{a} \div \sqrt[3]{a^2}$.
49. $\sqrt[3]{2x} \div \sqrt{x^3}$.
50. $\sqrt[5]{y^4} \div \sqrt[7]{2y^3}$.
51. $\sqrt[11]{b^5c^9} \div \sqrt[7]{b^4c^3}$

52. $(3+2\sqrt{5})(2-\sqrt{5})$.

53. $(8+3\sqrt{2})(2-\sqrt{2})$.

54. $(5+2\sqrt{3})(3-5\sqrt{3})$.

55. $(3-\sqrt{6})(6-3\sqrt{6})$.

56. $(\sqrt{5-\sqrt{9}})(\sqrt{5+\sqrt{9}})$.

57. $(\sqrt{3-\sqrt{2}})(\sqrt{3+\sqrt{2}})$.

58. $(\sqrt{9-\sqrt{17}})(\sqrt{9+\sqrt{17}})$.

59. $(\sqrt{6+\sqrt{11}})(\sqrt{6-\sqrt{11}})$.

60. $(\sqrt[3]{9+\sqrt{17}})(\sqrt[3]{9-\sqrt{17}})$.

61. $(\sqrt[3]{12+\sqrt{19}})(\sqrt[3]{12-\sqrt{19}})$.

EXERCISE 131.

POWERS AND ROOTS OF SURDS.

340. Any power of a surd can be expressed as a single surd by means of the principle of Art. 323. Thus:

(1) Square $\sqrt[3]{2}$.

$$(\sqrt[3]{2})^2=\sqrt[3]{2^2} \qquad \text{by [5].}$$
$$=\sqrt[3]{4}$$

(2) Cube $3\sqrt{2}$.

$$(3\sqrt{2})^3=3^3(\sqrt{2})^3 \quad \text{by law } D \text{ for indices.}$$
$$=27\sqrt{2^3} \qquad \text{by [5].}$$
$$=54\sqrt{2} \qquad \text{by exercise 124.}$$

The result should always be left in its simplest form.

341. Any root of a surd can be expressed as a single surd by means of the principle of Art. 321. Thus:

(1) Find square root of $\sqrt[3]{4}$.

$$\sqrt{\sqrt[3]{4}}=\sqrt[6]{4}=\sqrt[3]{\sqrt{4}} \qquad \text{by [3].}$$
$$=\sqrt[3]{2}$$

(2) Find cube root of $\frac{8}{9}\sqrt{3}$.

$$\sqrt[3]{\tfrac{8}{9}\sqrt{3}} = 2\sqrt[3]{\tfrac{1}{9}\sqrt{3}} \qquad \text{by exercise 124.}$$

$$= 2\sqrt[3]{\sqrt{\tfrac{3}{81}}} = 2\sqrt[3]{\sqrt{\tfrac{1}{27}}} \quad \text{by Ex. 125.}$$

$$= 2\sqrt{\sqrt[3]{\tfrac{1}{27}}} \qquad \text{by [3].}$$

$$= 2\sqrt{\tfrac{1}{3}} = \tfrac{2}{3}\sqrt{3} \qquad \text{by exercise 126.}$$

(3) Find the tth root of $a\sqrt[r]{b}$.

$$\sqrt[t]{a\sqrt[r]{b}} = \sqrt[t]{\sqrt[r]{a^r b}} \qquad \text{by exercise 125.}$$

$$= \sqrt[rt]{a^r b} \qquad \text{by [3].}$$

342. This last process is a *general* one, but if for any particular values of a, b, r, and t this result should not happen to be in its simplest form, it should be so reduced.

EXAMPLES.

Express each of the following as a surd in simplest form:

1. $(\sqrt{5})^4$.
2. $(\sqrt[3]{3})^4$.
3. $(2\sqrt{2})^3$.
4. $(\frac{1}{2}\sqrt[6]{2})^2$.
5. $(\frac{3}{2}\sqrt[3]{\frac{4}{9}})^3$.
6. $(-\sqrt[4]{8})^3$.
7. $(\frac{2}{3}\sqrt[3]{a^2})^2$.
8. $(a\sqrt[3]{\frac{3}{5}})^4$.
9. $\sqrt[3]{2\sqrt{8}}$.
10. $\sqrt{\sqrt[5]{36}}$.
11. $\sqrt[3]{3\sqrt{3}}$.
12. $\sqrt[3]{\sqrt{\frac{1}{5}}}$.
13. $\sqrt[3]{\frac{1}{4}\sqrt{\frac{1}{2}}}$.
14. $\sqrt[4]{\sqrt[9]{16}}$.
15. $\sqrt[3]{3\sqrt[4]{9a^{18}}}$.
16. $\sqrt[7]{128\sqrt[5]{243b^{70}}}$
17. $\sqrt[4]{\frac{1}{16}b^2\sqrt[3]{9b^2c^2}}$
18. $\sqrt[3]{\frac{2}{3}\sqrt{\frac{81}{5}}}$.
19. $2\sqrt[5]{\sqrt[t]{a}} - 4\sqrt[st]{a} + 2\sqrt[t]{\sqrt[s]{a}} = ?$

EXERCISE 132.

RATIONALIZATION OF EXPRESSIONS CONTAINING QUADRATIC SURDS.

343. To Rationalize an expression is to perform an operation upon it that will free the expression of surds. Thus, the binomial quadratic surd $3+\sqrt{2}$ is rationalized when multiplied by $3-\sqrt{2}$, for the product is $9-2$ or 7, which is rational.

344. Any multiplier which, when applied to an irrational expression, will free the expression from surds, is called a **Rationalizing Factor.** Thus, $3-\sqrt{2}$ is a rationalizing factor for the binomial surd $3+\sqrt{2}$.

345. It is often convenient to perform an operation upon both terms of a fraction so as to render either the numerator or the denominator rational. It is sufficient for present purposes to show how this may be done when all the surds are of the second order. The following are examples.

(1) Rationalize the denominator of $\frac{2}{2+\sqrt{2}}$.

Multiplying numerator and denominator by $2-\sqrt{2}$, we get

$$\frac{2}{2+\sqrt{2}}=\frac{2(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}=\frac{4-2\sqrt{2}}{4-(\sqrt{2})^2}$$
$$=\frac{4-2\sqrt{2}}{4-2}=2-\sqrt{2}.$$

(2) Rationalize the denominator of $\frac{3}{\sqrt{5}-\sqrt{2}}$.

Multiplying numerator and denominator by $\sqrt{5}+\sqrt{2}$, we get

$$\frac{3}{\sqrt{5}-\sqrt{2}}=\frac{3(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$$
$$=\frac{3(\sqrt{5}+\sqrt{2})}{5-2}=\sqrt{5}+\sqrt{2}.$$

346. This work is based on the very evident principle that *any binomial quadratic surd is made rational by multiplying it by itself with the sign of one of its terms changed, for the product is then the difference of two squares.*

347. Considerable labor is often saved in computing the value of an irreducible fraction if we first rationalize the denominator. Thus, to compute the value of

$$\frac{3}{\sqrt{5}-\sqrt{2}}$$

to five decimal places, the two square roots must be taken to at least five decimal places, and the *quotient of* 3 *divided by the difference of these roots must be found.* This division by a five-place number will be avoided if we first rationalize the denominator, for the result is $\sqrt{5}+\sqrt{2}$, the value of which is found without the long division of the former method.

EXAMPLES.

Rationalize the denominator in each of the following:

1. $\frac{2}{3+\sqrt{2}}$.

2. $\frac{7}{1-\sqrt{5}}$.

3. $\frac{9}{5-\sqrt{10}}$.

4. $\frac{4}{6-2\sqrt{3}}$.

5. $\frac{11}{\sqrt{8}-\sqrt{7}}$.

6. $\frac{6}{\sqrt{12}+\sqrt{5}}$.

7. $\frac{2}{\sqrt{3}+\sqrt{2}}$.

8. $\frac{3}{\sqrt{10}-\sqrt{6}}$.

9. $\frac{2\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$.

10. $\frac{21\sqrt{3}}{4\sqrt{3}-3\sqrt{2}}$.

11. $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

12. $\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$.

13. $\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$.

$$\frac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}=\frac{(1-\sqrt{2}+\sqrt{3})(1+\sqrt{2}+\sqrt{3})}{([1+\sqrt{2}]-\sqrt{3})([1+\sqrt{2}]+\sqrt{3})}=\frac{1+2\sqrt{3}+3-2}{1+2\sqrt{2}+2-3}=\frac{\sqrt{2}+\sqrt{6}}{2}$$

14. $\frac{30}{2-\sqrt{3}+\sqrt{5}}$.

15. $\frac{2+\sqrt{6}-\sqrt{2}}{2-\sqrt{6}+\sqrt{2}}$.

16. Find, in the shortest way, the value of $\frac{1}{3+2\sqrt{2}}$ to four places; given $\sqrt{2}=1.4142+$

EXERCISE 133.

RATIONALIZATION OF EQUATIONS.

348. If the unknown number in an equation appears under a radical sign, the equation must first be rationalized before the value of the unknown number can be found. This is illustrated by the following examples.

(1) Solve $\sqrt{5x}=20.$

Squaring both sides, we get

$$5x=400,$$

whence $$x=80.$$

(2) Solve $3\sqrt{4x-8}=\sqrt{13x-3}.$

Squaring both sides, we get

$$9(4x-8)=13x-3.$$

Transposing and uniting,

$$23x=69,$$

whence $$x=3.$$

(3) Solve $\sqrt{x+9}=5\sqrt{x}-3$

Squaring both sides, we get

$$x+9=25x-30\sqrt{x}+9.$$

Transposing rational terms to one side and irrational terms to the other, we get

$$24x=30\sqrt{x}$$

Dividing by 6 and squaring,

$$16x^2=25x.$$

Solving this quadratic, $x=\frac{25}{16}$ or 0.

(4) Solve $\sqrt{32+x}=16-\sqrt{x}.$

Squaring both sides,

$$32+x=256-32\sqrt{x}+x.$$

Transposing, uniting, and dividing by 32,

$$\sqrt{x}=7,$$

whence $$x=49.$$

EXAMPLES.

Solve each of the following equations:

1. $\sqrt[3]{2x}=4$.
2. $\sqrt[4]{8y}=2$.
3. $\sqrt{x+4}=4$.
4. $\sqrt{2x+6}=4$.
5. $\sqrt{10x+16}=5$.
6. $\sqrt{2x+7}=\sqrt{5x-2}$.
7. $14+\sqrt[3]{4x-40}=10$.
8. $\sqrt{16x+9}=4\sqrt{4x}-3$
9. $\sqrt{\frac{15}{4}+x}=\frac{3}{2}+\sqrt{x}$.
10. $\sqrt{x}-\sqrt{x-5}=\sqrt{5}$.
11. $\sqrt{x-7}=\sqrt{x-14}+1$.
12. $\sqrt{x-7}=\sqrt{x+1}-2$.
13. $x=7-\sqrt{x^2-7}$.
14. $\sqrt{x+20}-\sqrt{x-1}-3=0$.
15. $\frac{x-1}{\sqrt{x}-1}=\frac{\sqrt{x}+1}{x-3}$.
16. $\frac{3-x}{2-\sqrt{x}}=\frac{2+\sqrt{x}}{4-x}$.
17. $\sqrt{x+3}+\sqrt{3x-2}=7$.
18. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}$.

CHAPTER XVIII.

RATIO, PROPORTION AND VARIATION.

EXERCISE 134

RATIO.

349. The relative magnitude of two numbers or quantities, measured by the number of times that the first contains the second, is called the **Ratio** of the two numbers or quantities.

Thus, 12 contains 3 just 4 times, hence the ratio of 12 to 3 is 4 or $\frac{12}{3}$. And similarly if a and b stand for *any* two numbers the ratio of a to b is $\frac{a}{b}$.

350. We may speak of the ratio of two *quantities* of the same kind as well as the ratio of two *numbers*. Thus, 12 feet contains 3 feet just 4 times, hence the ratio of 12 feet to 3 feet is 4 or $\frac{12}{3}$.

The student should notice that the ratio of 12 feet to 3 feet is $\frac{12}{3}$, not $\frac{12 \text{ feet}}{3 \text{ feet}}$, for division proper cannot be performed except when the divisor is a *number*. Division implies separating into a certain number of equal parts. For example, a stick 12 feet long may be sawed into 3 equal pieces, and one of these pieces is 4 feet long; so 12 feet divided by 3 is 4 feet. But while a stick 12 feet long may be divided by 3 but not by 3 *feet*, still a stick 12 feet long may be *measured by* a stick 3 feet long. This *measured by* is often confused with *division* proper.

The ratio of 12 feet to 3 feet is not 12 feet divided by 3 feet, but it is 12 divided by 3.

The ratio of *any* two quantities *of the same kind* may be looked upon as the number of units in the first divided by the number of units in the second. Plainly, quantities which are not of the same

kind cannot have any ratio, for one cannot possibly be measured by the other, nor can one be contained at all in the other. For example, ten miles cannot be measured by two quarts, nor can two quarts be contained any number of times in ten miles.

351. The ratio of a to b is denoted in either of two ways: *first*, by writing the a before the b with a colon between them, thus, $a : b$; *second*, by a fraction in which a is the numerator and b is the denominator, thus, $\frac{a}{b}$.

Whichever way the ratio is written, it is read "the ratio of a to b," or simply "a to b."

352. In either way of writing the ratio of a to b, a is called the **Antecedent** or **First Term**, and b is called the **Consequent** or **Second Term.**

353. Since a ratio is the quotient obtained by dividing the number of units in the antecedent by the number of units in the consequent, it follows that the properties of ratios may be obtained immediately from the properties of fractions.

354. Since a fraction may be multiplied either by multiplying the numerator or dividing the denominator, it follows that *a ratio may be multiplied either by multiplying the antecedent or by dividing the consequent.*

355. Since a fraction may be divided either by dividing the numerator or multiplying the denominator, it follows that *a ratio may be divided either by dividing the antecedent or by multiplying the consequent.*

356. Since a fraction remains unchanged in value when both numerator and denominator are multiplied or

divided by the same number, it follows that *a ratio remains unchanged in value when both antecedent and consequent are multiplied or divided by the same number.*

357. If the numerator of a fraction is greater than the denominator, the fraction is greater than 1; therefore, *if the antecedent of a ratio is greater than the consequent, the ratio is greater than* 1.

358. If the numerator of a fraction is less than the denominator, the fraction is less than 1; therefore, *if the antecedent of a ratio is less than the consequent, the ratio is less than* 1.

359. If the numerator and denominator of a fraction are equal to each other, the fraction is equal to 1; therefore, *if the antecedent and consequent of a ratio are equal to each other, the ratio is equal to* 1.

360. Theorem. *A ratio which is greater than* 1 *is decreased by increasing both antecedent and consequent by the same amount.*

Let $\frac{a}{b}$ be a ratio which is greater than 1; then $a>b$. Now form a new ratio by increasing the antecedent and consequent by the same amount, x. The new ratio is

$$\frac{a+x}{b+x}.$$

If we multiply antecedent and consequent of the original ratio by $b+x$, we get

$$\frac{a}{b}=\frac{ab+ax}{b^2+bx}. \tag{1}$$

If we multiply antecedent and consequent of the new ratio by b, we get

$$\frac{a+x}{b+x}=\frac{ab+bx}{b^2+bx}. \tag{2}$$

Now, as $a>b$, it is plain that $ax>bx$, and hence $ab+ax>ab+bx$. Therefore,

$$\frac{ab+ax}{b^2+bx}>\frac{ab+bx}{b^2+bx}.$$

Therefore, from (1) and (2),

$$\frac{a}{b}>\frac{a+x}{b+x},$$

which is what was to be proved.

Since $a>b$, it is plain that $a+x>b+x$. Therefore

$$\frac{a+x}{b+x}>1.$$

Therefore, as each ratio $\frac{a}{b}$ and $\frac{a+x}{b+x}$ is greater than 1, and as $\frac{a}{b}>\frac{a+x}{b+x}$, it follows that $\frac{a+x}{b+x}$ *is nearer the value* 1 *than* $\frac{a}{b}$ *is.*

361. Theorem. *A ratio which is less than* 1 *is increased by increasing both antecedent and consequent by the same amount.*

Let $\frac{a}{b}$ be a ratio which is less than 1; then $a<^*b$. Now form a new ratio by increasing the antecedent and consequent by the same amount, x. The new ratio is

$$\frac{a+x}{b+x}.$$

If we multiply antecedent and consequent of the original ratio by $b+x$, we get

$$\frac{a}{b}=\frac{ab+ax}{b^2+bx}. \qquad (1)$$

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* This notation denotes that a is less than b. Whenever this symbol is used the point of the angle is toward the lesser and the opening toward the greater number. See note, page 291.

If we multiply antecedent and consequent of the new ratio by b, we get

$$\frac{a+x}{b+x}=\frac{ab+bx}{b^2+bx}. \qquad (2)$$

Now, as $a<b$, it is plain that $ax<bx$, and hence $ab+ax<ab+bx$. Therefore,

$$\frac{ab+ax}{b^2+bx}<\frac{ab+bx}{b^2+bx}.$$

Therefore, from (1) and (2),

$$\frac{a}{b}<\frac{a+x}{b+x},$$

which is what was to be proved.

Since $a<b$, it is plain that $a+x<b+x$. Therefore,

$$\frac{a+x}{b+x}<1.$$

Therefore, as each ratio $\frac{a}{b}$ and $\frac{a+x}{b+x}$ is less than 1, and as $\frac{a}{b}<\frac{a+x}{b+x}$, it follows that $\frac{a+x}{b+x}$ *is nearer the value* 1 *than* $\frac{a}{b}$ *is.*

This last statement, together with the last statement in the previous article, shows that *any ratio* (except the ratio 1) *is made more nearly the value* 1 *by increasing both antecedent and consequent by the same amount.*

362. If from two given ratios, $\frac{a}{b}$ and $\frac{c}{d}$ or $a:b$ and $c:d$, we form another ratio by multiplying the antecedents together for a new antecedent and the consequents together for a new consequent, we get $\frac{ac}{bd}$ or $ac:bd$, which is said to be **Compounded** of the given ratios $a:b$ and $c:d$.

363. If $x>1$ it follows that $xy>y$, or, expressed in words, if the multiplier is greater than 1 the product is greater than the multiplicand.

Therefore, if $\frac{c}{d}>1$, $\frac{a}{b}\frac{c}{d}>\frac{a}{b}$. Also, if $\frac{a}{b}>1$, $\frac{a}{b}\frac{c}{d}>\frac{c}{d}$.

But in each of these cases $\frac{a}{b}\frac{c}{d}$ or $\frac{ac}{cd}$ is the ratio compounded of the ratios $\frac{a}{b}$ and $\frac{c}{d}$.

Therefore, *if a ratio be compounded of two ratios each of which is greater than* 1 *the result is greater than either of the given ratios.*

364. In a similar way it may be shown that *if a ratio be compounded of two ratios each of which is less than* 1 *the result is less than either of the given ratios.*

365. If $x<1$ it follows that $xy<y$, or, expressed in words, if the multiplier is less than 1 the product is less than the multiplicand.

Therefore, if $\frac{c}{d}<1$, $\frac{a}{b}\frac{c}{d}<\frac{a}{b}$, and it was before shown that if $\frac{a}{b}>1$, $\frac{a}{b}\frac{c}{d}>\frac{c}{d}$.

Therefore, *if a ratio be compounded of two given ratios, one of which is greater than* 1 *and the other less than* 1, *the result is intermediate in value between the two given ratios.*

PROBLEMS.

1. Arrange the ratios $2:3$, $3:5$, $5:8$ in the order of magnitude.

2. Which is nearer unity $2:3$ or $8:9$?

3. Which is nearer unity $2:3$ or $2+x:3+x$?

4. For what value of x will the ratio $8+x:36+x$ be equal to the ratio $1:3$?

5. What must be added to each term of the ratio 9 : 16 to produce the ratio 3 : 4?

6. What must be subtracted from each term of the ratio 3 : 5 to produce the ratio 5 : 9?

7. A certain ratio will become equal to $\frac{1}{3}$ when 1 is subtracted from each of its terms, and equal to $\frac{3}{4}$ when 9 is added to each of its terms. Find the ratio.

8. Find two numbers such that if each is increased by 1 the results have the ratio 2 : 3, and if each is increased by 7 the results have the ratio 5 : 6.

9. Find two numbers such that if the first is increased by 2 and the second decreased by 2 the results have the ratio 6 : 5, but if the first is decreased by 2 and the second increased by 2 the results have the ratio 4 : 7.

10. A rectangle is 39 feet long and 36 feet wide. Express the ratio of the length to the breadth.

11. What is the ratio of 12 lbs. 8 oz. to 21 lbs. 14 oz.?

12. What is the ratio of 5 ft. to 6 ft. 3 in.?

13. Two rectangular fields have the same area. The length of the first is 180 feet and of the second 150 feet. What is the ratio of their widths?

14. The areas of two rectangular rooms have the ratio 2 : 3. The length of the first is 12 feet, and of the second 20 feet. What is the ratio of their widths?

15. Two numbers are in the ratio of 3 : 4, and the sum of their squares is 400. What are the numbers?

16. Two equal sums of money are on interest; the first runs 8 months at 7 per cent., the second runs 9 months at 6 per cent. The interest in the first case has what ratio to the interest in the second case?

17. Divide the number 10 into two such parts that the squares of these parts shall have the ratio 9 : 4.

18. A train of cars travels 140 miles in $3\frac{1}{2}$ hours, and another train travels 240 miles in 5 hours. What is the ratio of their rates of speed?

19. Show that the ratio $a : b$ is equal to $(a+x)^2 : (b+x)^2$ if $x^2 = ab$.

20. What must be the value of x in order that the ratio $6 : x$ may equal $5-x$?

EXERCISE 135.

PROPORTION.

366. The expression of equality which exists between two ratios is called a **Proportion.** Thus, $a : b = c : d$ is a proportion. In this case the four numbers or quantities are said to be **in proportion.**

367. Sometimes the proportion is expressed as an ordinary equation in fractions, thus, $\frac{a}{b} = \frac{c}{d}$, and sometimes by the notation $a : b :: c : d$. However written, the proportion is read, "a is to b as c is to d."

368. In the proportion $a : b = c : d$ the letters a, b, c, and d are called the **Terms** of the proportion. The first and fourth terms are called the **Extremes,** and the second and third terms are called the **Means.**

369. If a, b, c, and d are proportional, we have the proportion $a : b :: c : d$, or, written in the fractional form, $\frac{a}{b} = \frac{c}{d}$. Now, multiplying each member by bd, we get

$$ad = bc.$$

That is, *the product of the means equals the product of the extremes.*

370. If we have given the equation

$$ad=bc,$$

then dividing each member by bd, we get

$$\frac{ad}{bd}=\frac{bc}{bd} \text{ or } \frac{a}{b}=\frac{c}{d} \text{ or } a:b::c:d.$$

Hence, *if the product of two numbers equals the product of two other numbers, the numbers are in proportion*, or, in other words, *if the product of two numbers equals the product of two other numbers, the factors of one product may be taken for the extremes and the factors of the other product may be taken for the means of a proportion.*

371. From the equation $ad=bc$ we may infer either

$$a:b=c:d \quad (1)$$

or $$a:c=b:d \quad (2)$$

or $$b:a=d:c \quad (3)$$

or $$b:d=a:c \quad (4)$$

and therefore from (1) we may infer either (2) or (3) or (4).

The proportion (2) is said to be deduced from (1) by **Alternation.** The proportion (3) is deduced from (1) by **Inversion.** The proportion (4) is really the same as (2) except that the two members have changed places; it may be deduced from (3) by alternation.

372. If $a:b=c:d$, then writing in fractional form,

$$\frac{a}{b}=\frac{c}{d}$$

Adding 1 to each member,

$$\frac{a}{b}+1=\frac{c}{d}+1,$$

or in another form, $$\frac{a+b}{b}=\frac{c+d}{d},$$

or in the common form of proportion,

$$a+b:b=c+d:d.$$

This last proportion, $a+b:b=c+d:d$, is said to be derived from the proportion $a:b=c:d$ by **Composition.**

373. If $a : b = c : d$, then $\frac{a}{b} = \frac{c}{d}$, and subtracting 1 from each member,

$$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or in another form,

$$\frac{a-b}{b} = \frac{c-d}{d},$$

or in the common form of proportion,

$$a-b : b = c-d : d.$$

This last proportion, $a-b : b = c-d : d$, is said to be derived from the proportion $a : b = c : d$ by **Division.**

374. If $a : b = c : d$, then by composition,

$$\frac{a+b}{b} = \frac{c+d}{d}, \tag{1}$$

and by division,

$$\frac{a-b}{b} = \frac{c-d}{d}. \tag{2}$$

Divide (1) and (2) member by member and we get

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

or written in the ordinary form of a proportion,

$$a+b : a-b = c+d : c-d.$$

This last proportion, $a+b : a-b = c+d : c-d$, is said to be derived from the proportion $a : b = c : d$ by **Composition and Division.**

375. *The products of corresponding terms of two or more sets of proportional numbers are proportional.*

Let $a : b = c : d$

and $e : f = h : k$

and $n : r = s : t$

Writing each of these proportions as an equation in fractions, we have

$$\frac{a}{b}=\frac{c}{d}$$

$$\frac{e}{f}=\frac{h}{k}$$

$$\frac{n}{r}=\frac{s}{t}$$

and from these equations by multiplication we obtain

$$\frac{a}{b}\frac{e}{f}\frac{n}{r}=\frac{c}{d}\frac{h}{k}\frac{s}{t}$$

or

$$\frac{aen}{bfr}=\frac{chs}{dkt}$$

or writing this in the ordinary notation of proportion, we have $aen : bfr = chs : dkt$,
which is what was to be proved.

376. *Like powers or like roots of proportional numbers are proportional.*

Let $a : b = c : d.$

Writing this proportion as an equation in fractions, we obtain

$$\frac{a}{b}=\frac{c}{d}, \tag{1}$$

and raising each member to the n th power, we have

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}, \tag{2}$$

or writing this in the ordinary notation of proportion, we have

$$a^n : b^n = c^n : d^n. \tag{3}$$

Again, taking the n th root of each number of (1), we have

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}} \tag{4}$$

or writing in the ordinary notation of proportion, we have

$$a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}. \tag{5}$$

Equations (3) and (5) are those which were to be proved.

377. In the proportion $a : b = b : c$, b is called a **Mean Proportional** between a and c, and c is called a **Third Proportional** to a and b.

If in this proportion we write the product of the means equal to the product of the extremes, we get

$$b^2 = ab,$$

or extracting the square root of each member, we get

$$b = \sqrt{ab}.$$

Therefore, *a mean proportional between two numbers is equal to the square root of their product.*

378. A **Continued Proportion** is a series of equal ratios. Thus, $a : b = c : d = e : f$, etc.

379. In any continued proportion *any antecedent is to its corresponding consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let the continued proportion be written as several equal fractions, thus:

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

and let r be the common value of each of these fractions, then

$$\frac{a}{b} = r \text{ or } a = br,$$

$$\frac{c}{d} = r \text{ or } c = dr,$$

$$\frac{e}{f} = r \text{ or } e = fr.$$

Therefore, by addition,

$$a + c + e = br + dr + fr = (b + d + f)r.$$

Hence, by dividing by $b + d + f$,

$$\frac{a + c + e}{b + d + f} = r = \frac{a}{b},$$

which is what was to be proved.

Examples and Problems.

1. Write two proportions from the equation $5 \times 8 = 4 \times 10$
2. Write two proportions from the equation $ab = cd$.
3. Write two proportions from the equation $ab = ac$.
4. Write two proportions from the equation $x^2 = yz$.
5. Write two proportions from the equation $(a+b)^2 = nr$
6. Write two proportions from the equation $3n^2r = 4pq$.
7. Form a proportion with the numbers 3, 5, 21, 35.
8. Form a proportion with the numbers 3, 20, 90, 600.
9. Form a proportion with the numbers 3, 6, 20, 40.
10. What must x stand for in order that $x : 3 = 6 : 9$ may be a true proportion?
11. What is the value of x if $5 : x = 6 : 12$?
12. What is the value of x if $12 : 15 = x : 35$?
13. What is the value of x if $8 : 10 = 90 : x$?
14. What is the value of x if $x : 6 = x-1 : x+11$?
15. What is the value of x if
$$\frac{15(a+b)}{14(a-b)} : \frac{10(ab)^2}{21(ab)^2} = \frac{a-b}{a+b} : x ?$$
16. What is the value of x if
$$\left(\frac{a^3-b^3}{a-b} - ab\right) : \left(\frac{a^3+b^3}{a+b} + ab\right) = 1 : x ?$$
17. What is the value of x if
$x : (5n-6r) = (3n^2+2nr-8r^2) : (5n^2+4nr-12r^2)$?
18. What is the value of x if
$$(a+b) : x = (a^2+ab+b^2) : (a^3-b^3) ?$$
19. If $a : b = c : d$, prove $a : mb = c : md$.
20. If $a : b = c : d$, prove $na : rb = nc : rd$.
21. If $a : b = c : d$, prove
$$(na+rb) : (na-rb) = (nc+rd) : (nc-rd).$$

22. If $a:b=c:d$, prove $na:rb=\frac{c}{r}:\frac{d}{n}$.

23. If $a:b=c:d$, prove that $a:a+c=ab:a+b+c+d$.

24. If $a:b=c:d$, prove that
$a^2+ab+b^2:a^2-ab+b^2=c^2+cd+d^2\cdot c^2-cd+d^2$.

25. Prove that $a:b=c:d$, if
$(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$.

EXERCISE 136.

VARIATION.

380. Thus far whenever a letter has been used to stand for a number, it has always represented the *same* number throughout the same problem; or in other words, the value of the letter has always remained the same in the same problem. But sometimes it is necessary to consider quantities which *change* in value in the same problem, and then the letter used in connection with such a quantity may stand for one number at one instant and another number at another instant in the same problem. For example, if a train of cars travels at the rate of 40 miles an hour, of course the number of miles traveled depends upon the time of traveling. Now we may take a letter, say x, to represent the number of miles the train travels, and another letter, say y, to represent the number of hours the train travels; then plainly we have

$$x=40y \text{ or } \frac{x}{y}=40.$$

Here x and y stand for numbers which are *not* the same at one time as at another; or in other words, x and y *change* in value, but however much or little they change, the *ratio* of these remains *unchanged*, or, as it is usually expressed, the *ratio remains constant.*

381. One number or quantity is said to **Vary Directly** as another when the number of units in the first is equal to the number of units in the second multiplied or divided by some constant number, or what is the same thing, when the ratio of the number of units in the first to the number of units in the second is some constant number.

In the above illustration the distance traveled is said to vary directly as the time occupied, or the number of miles traveled varies directly as the number of hours occupied.

382. The word *directly* is often omitted, and we say simply, that one number or quantity *varies* as another. The same idea is often expressed by saying that the second number or quantity is **Proportional to** the first number or quantity. In the above illustration the distance traveled is *proportional to* the time occupied, or the number of miles traveled is *proportional to* the number of hours occupied.

383. If x is proportional to y then $\frac{x}{y}=c$ where c is some constant, *i. e.* some number that remains unchanged. Now, if we represent some particular value of x by x_1 and the corresponding particular value of y by y_1, then $\frac{x_1}{y_1}=c$ and from this it is easily seen that when one number is proportional to another, we know the constant by which one of the numbers must be multiplied to produce the other, if we know any corresponding *particular* values of the two numbers. For example, if we know that a traveler is walking uniformly, *i. e.* at the same rate, then we know that the number of miles traveled is proportional to the number of hours occupied, hence if we

represent the number of miles traveled by x and the number of hours occupied by y, we have

$$x=cy \text{ or } \frac{x}{y}=c.$$

Now if we further know that the traveler walks 15 miles in 5 hours, we have $c=\frac{15}{5}=3$.

From this we see that 3 may be written in place of c, and hence we know $x=3y$ or $\frac{x}{y}=3$.

384. One number or quantity is said to **Vary Inversely** as another when the first is equal to some constant divided by the second. Thus, x varies inversely as y, if $x=\frac{c}{y}$ where c is some constant.

385. If in the equation $x=\frac{c}{y}$ we multiply each number by y, we get $xy=c$.

Therefore, we say that *one number varies inversely as a second when the product of the two numbers is some constant.* For example, the time occupied in traveling a certain distance *varies inversely* as the rate of speed, or the number of hours occupied in traveling a certain number of miles *varies inversely* as the number of miles traveled per hour.

386. Instead of saying that one number varies inversely as another, the same idea is often expressed by saying that one number is **Reciprocally Proportional** to another.

387. One number **Varies Jointly** as two others when the first is equal to some constant multiplied by the product of the other two. Thus, if $x=cyz$, or $\frac{x}{yz}=c$, where c is some constant, x is said to vary jointly as y and z. The same idea is often expressed by saying that one number is **Proportional to the Product** of two others.

388. One number is said to vary as the square of another when the first is some constant multiplied by the square of the second, or when the first divided by the square of the second is some constant. Thus, if $x=cy^2$, or $\frac{x}{y^2}=c$, where c is some constant, x is said to vary as y^2.

The same idea is often expressed by saying that one number is **Proportional to the Square** of another.

389. One number is said to **Vary Directly** as a second and **Inversely** as a third when the first is equal to some constant multiplied by the ratio of the second to the third. Thus, if $x=c\frac{y}{z}$, x is said to vary directly as y and inversely as z.

The same idea is often expressed by saying that the first number is **Directly Proportional** to the second and **Inversely Proportional** to the third.

Examples and Problems.

1. If x is proportional to y, and $x=45$ when $y=3$, find the value of x when $y=15$.

2. If x is inversely proportional to y, and $x=1$ when $y=6$, find the value of x when $y=15$.

3. Form a proportion with the two sets of values of x and y in problem 2.

4. If x varies as y^2, and $x=1$ when $y=5$, find the value of x when $y=15$.

5. Form a proportion with the two sets of values of x and y^2 in problem 4.

6. If x varies jointly as y and z, and $x=20$ when $y=2$ and $z=3$, find the value of x when $y=3$ and $z=6$.

7. If x varies inversely as the square of y, and $x=1$ when $y=10$, find the value of x when $y=5$.

8. If x is directly proportional to y and inversely proportional to z, and $x=20$ when $y=6$ and $z=4$, find the value of x when $y=3$ and $z=10$.

9. If x varies as y, prove that x^3 varies as y^3.

10. If x is inversely proportional to y and y is inversely proportional to z, prove that x is proportional to z.

11. If x is proportional to z and y is also proportional to z, prove that xy is proportional to z^2, also that x^2+y^2 is proportional to z^2.

12. If $3x+7y$ is proportional to $3x+13y$ and $x=5$ when $y=3$, find the ratio of x to y and thus show that x varies as y.

13. The number of feet a body falls is proportional to the square of the number of seconds occupied in falling. Knowing that a body falls 16 feet the first second, find how many feet it will fall in 5 seconds.

14. With the same supposition as in the last example, find the height of a tower from which a stone dropped from the summit, reaches the ground in $3\frac{3}{4}$ seconds.

15. The weight of a metal ball is proportional to the cube of the radius, and a ball whose radius is 2 inches weighs 10 pounds, what is the weight of a ball whose radius is 5 inches?

16. If a heavier weight draw up a lighter one by means of a cord passed over a fixed wheel, the number of feet passed over by each weight in any given time varies directly as the difference of the weights, and inversely as the sum of the weights. If 10 pounds draw up 6 pounds 16 feet in 2 seconds, how high will 14 pounds draw 10 pounds in 2 seconds?

CHAPTER XIX.

PROGRESSIONS.

EXERCISE 137.

ARITHMETICAL PROGRESSIONS.

390. An **Arithmetical Progression** is a series of terms such that each term differs from the immediately preceding term by a fixed number, called the **Common Difference.** The following are examples of arithmetical progressions:

(1) 2, 4, 6, 8, 10. (3) $2\frac{1}{2}$, $3\frac{3}{4}$, 5, $6\frac{1}{4}$, $7\frac{1}{2}$.

(2) 31, 26, 21, 16. (4) $(x-y)$, x, $(x+y)$.

(5) a, $(a+d)$, $(a+2d)$, $(a+3d)$.

391. The first and last terms of any given progression are called the **Extremes,** and the other terms are called the **Means.**

1. The first term of an arithmetical progression is 3 and the common difference is 2. What is the 5th term? What is the 10th term?

2. The first term of an arithmetical progression is 5 and the common difference is 3. What is the 4th term? What is the 7th term?

3. In the last progression, how many times must the common difference be added to 5 to produce the 4th term? to produce the 7th term? to produce the nth term?

4. The first term of an arithmetical progression is 19 and the common difference -4. What is the third term? What is the fifth term? What is the nth term?

392. The nth Term of any Arithmetical Progression. It is usual to represent the first term of an arithmetical progression by a, the common difference by d, and the nth term by l. With this notation we may represent any arithmetical progression by

No. of term:	1	2	3	4	5 . . .
Progression:	a,	$(a+d)$,	$(a+2d)$,	$(a+3d)$,	$(a+4d)$

We notice that the coefficient of d in the 2d term is 1, in the 3d term is 2, in the 4th term is 3, and, by the nature of the progression, the coefficient of d in any term is 1 less than the number of that term. Therefore, the nth term in this progression will be

$$a+(n-1)d,$$

or, representing the nth term by l, we have the formula

$$\boldsymbol{l=a+(n-1)d.} \qquad [1]$$

393. Evidently the sum of an arithmetical progression is not changed if the order of the terms be reversed; thus,

$3+5+7+9+11$ may be written $11+9+7+5+3$

$2\frac{1}{2}+3\frac{1}{4}+4+4\frac{3}{4}$ may be written $4\frac{3}{4}+4+3\frac{1}{4}+2\frac{1}{2}$

in which case the first term becomes the last term, the last term becomes the first term, and the *common difference changes sign.*

394. When the common difference is positive the progression may be called an **Increasing Progression,** and when the common difference is negative the progression may be called a **Decreasing Progression.**

395. The Sum of n Terms of any Arithmetical Progression. If s stands for the sum of n terms of an arithmetical progression, evidently we may write the two

following equalities, the progressions being alike except written in reverse order:

$$s=a+(a+d)+(a+2d)+(a+3d)+\ldots+a+(n-1)d \quad (1)$$
$$s=l+(l-d)+(l-2d)+(l-3d)+\ldots+l-(n-1)d \quad (2)$$

Adding (1) and (2) together term for term, noticing that the terms containing the common difference nulify one another, we have

$$2s=(a+l)+(a+l)+(a+l)+(a+l)+\ldots+(a+l).$$

Since the number of terms in the original progression has been called n, we write the last equation

$$2s=n(a+l),$$

whence the formula for s,

$$s=\tfrac{1}{2}n(a+l). \qquad [2]$$

396. Formula [1] enables us to obtain the value of l when a, n, and d are given, or the value of a, when l, n, and d are given, or the value of d when l, a, and n are given, or the value of n when l, a, and d are given. Thus:

(1) Find the 20th term of $3+8+13+\ldots$

Here $a=3$, $d=5$, $n=20$, therefore

$$l=3+19\times 5=98.$$

(2) Find the number of terms in the progresssion $5+7+9+\ldots+37$.

Here $a=5$, $d=2$, $l=37$, whence

$$37=5+(n-1)2.$$

Solving for n, $$n=17.$$

(3) Find the common difference in a progression of 11 terms where the extremes are $\frac{1}{2}$ and $30\frac{1}{2}$.

Here $a=\frac{1}{2}$, $l=30\frac{1}{2}$ and $n=11$, whence

$$30\tfrac{1}{2}=\tfrac{1}{2}+(11-1)d.$$

Solving for d, $$d=3.$$

(4) Insert 3 arithmetical means between 5 and 21.

Here $n=5$, $a=5$, and $l=21$, whence

$$21=5+(5-1)d.$$

Solving for d, $d=4$.

Therefore, the means are 9, 13, and 17.

397. Formula [2] enables us to find any *one* of the numbers s, n, a and l, when the value of the other three are given. Thus:

(1) Find the sum of 10 terms of the progression where 5 is the first term and -58 the last term.

Here $a=5$, $n=10$, $l=-58$, whence

$$s=\tfrac{1}{2}\times 10(5-58.)$$

That is, $s=-265$.

(2) Find the number of terms in an arithmetical progression where the first term is 4, the last term 22, and the sum 91.

Here $a=4$, $l=22$ and $s=91$, whence

$$91=\tfrac{1}{2}\,n(4+22.)$$

Solving for n, $n=7$.

398. The two formulas

$$\left\{\begin{aligned} l&=a+(n-1)d && (1)\\ s&=\tfrac{1}{2}\,n(a+l) && (2)\end{aligned}\right.$$

contain five different letters, hence if any two of them stand for unknown numbers, and the values of the rest are given, the value of the two unknown numbers can be obtained by the solution of a system of two equations. Thus:

(1) Find the sum of an arithmetical progression where the last term is 149, the common difference 7, and the number of terms 22.

Here $l=149$, $d=7$ and $n=22$, whence

$$\left\{\begin{array}{ll} 149=a+(22-1)7 & (1) \\ s=\frac{1}{2}\times 22(a+149.) & (2) \end{array}\right.$$

From (1), $a=2$.

Substituting in (2) $s=1661$.

(2) Find the first term of an arithmetical progression of 21 terms, whose sum is 1197 and common difference is 4.

Here $n=21$, $s=1197$ and $d=4$, whence

$$\left\{\begin{array}{ll} l=a+(21-1)4 & (1) \\ 1197=\frac{1}{2}\times 21(a+l). & (2) \end{array}\right.$$

From (1), $l=a+80.$ (3)

From (2), $l=114-a.$ (4)

Whence, $l=97$ and $a=17$.

(3) Find the number of terms in a progression whose sum is 1095, the first term is 38 and the difference is 5.

Here $s=1095$, $a=38$ and $d=5$, whence

$$\left\{\begin{array}{ll} l=38+(n-1)5 & (1) \\ 1095=\frac{1}{2}n(38+l) & (2) \end{array}\right.$$

From (1), $l=33+5n.$ (3)

From (2), $2190=38n+nl.$ (4)

Substituting the value of l from (3) in (4) we get

$$2190=71n+5n^2. \quad (5)$$

Solving this quadratic equation, we find

$$n=15 \text{ or } -29.2.$$

The second result is inadmissable, since the number of terms cannot be either negative or fractional.

EXERCISE 138.

Examples and Problems.

Solve each of the following:

1. Given $a=7$, $d=4$, $n=13$; find l and s.
2. Given $a=\frac{1}{2}$, $d=6$, $n=30$; find l and s.
3. Given $a=9$, $l=162$, $n=52$; find s and d.

4. Given $a=57$, $l=30$, $n=19$; find s and d.
5. Given $l=242$, $d=21$, $n=12$; find a and s.
6. Given $l=16$, $d=-3$, $n=58$; find a and s.
7. Given $a=17$, $l=350$, $d=9$; find n and s.
8. Given $a=-28$, $l=28$, $d=7$; find n and s.
9. Given $a=1\frac{1}{2}$, $l=54$, $s=999$; find n and d.
10. Given $a=5$, $l=161$, $s=3320$; find n and d.
11. Given $a=3$, $n=50$, $s=3825$; find l and d.
12. Given $a=-45$, $n=31$, $s=0$; find l and d.
13. Given $l=49$, $n=19$, $s=503\frac{1}{2}$; find a and d.
14. Given $l=105$, $n=16$, $s=840$; find a and d.
15. Given $n=35$, $s=2485$, $d=3$; find a and l.
16. Given $n=25$, $s=-25$, $d=\frac{1}{2}$; find a and l.
17. Given $s=4784$, $a=41$, $d=2$; find l and n.
18. Given $s=624$, $a=9$, $d=4$; find l and n.
19. Given $s=278$, $d=5$, $l=77$; find a and n.
20. Given $s=1008$, $d=4$, $l=88$; find a and n.
21. What is the sum of the first 200 natural numbers?
22. What is the sum of the even numbers from 0 to 200?
23. What is the sum of the odd numbers from 1 to 200?
24. What is the sum of the first n even numbers?
25. What is the sum of the first n odd numbers?
26. Insert 9 arithmetical means between $-\frac{7}{8}$ and $+\frac{7}{8}$.
27. Sum the series $\sqrt{\frac{1}{2}}+\sqrt{2}+3\sqrt{\frac{1}{2}}+\ldots$ to 20 terms.
28. Sum the series $5-2-9-\ldots$ to 8 terms.
29. Insert 5 arithmetical means between 10 and 8.
30. Insert 4 arithmetical means between -2 and -16.
31. Sum $(a+b)^2+(a^2+b^2)+(a-b)^2$ to n terms.
32. Find the sum of the first 10 multiples of 3.

33. Find the sum of the first 50 multiples of 7.

34. Find the sum of the odd numbers between 200 and 300.

35. Find the arithmetical progression whose sum is 550 and whose middle term equals 50.

36. The sum of 25 successive terms of the progression $5+8+11+\ldots$ is 1025; what is the first term?

37. The sum of 10 terms of an arithmetical progression is 15, and the fifth term is 0; what is the first term?

38. How many terms of the progression $9+13+17+\ldots$ must be taken in order that the sum may equal 624?

39. We must take how many terms of the progression $\left(1+\frac{1}{h}\right)+\left(1+\frac{3}{h}\right)+\left(1+\frac{5}{h}\right)+\ldots$ in order that the sum may be $6h$?

40. How many terms must be taken from the commencement of the series $1+5+9+13+17$, etc., so that the sum of the 13 succeeding terms shall be 741?

41. The sum of the first three terms of an arithmetical progression is 15, and the sum of their squares is 83; find the common difference.

Let x=first term and y the common difference. Then

$$x+(x+y)+(x+2y)=15,$$

and

$$x^2+(x+y)^2+(x+2y)^2=83.$$

Another notation which is very convenient in a problem like this is: Represent the three terms by $x-y$, x, and $x+y$, whence we would write

$$(x-y)+x+(x+y)=15,$$

and

$$(x-y)^2+x^2+(x+y)^2=83.$$

42. There are two arithmetical progressions which have the same common difference; the first terms are 3 and 5 respectively, and the sum of seven terms of the the one is to the sum of seven terms of the other as 2 to 3. Determine the progressions.

43. The sum of three numbers in arithmetical progression is 12, and the sum of their squares is 66. Find the numbers.

44. The sum of three numbers in arithmetical progression is 33, and the sum of their squares is 461. What are the numbers?

EXERCISE 139.

GEOMETRICAL PROGRESSIONS.

399. A **Geometrical Progression** is a series of terms such that each term is the product of the preceding term by a fixed factor called the **Ratio.** The following are examples:

(1) 3, 6, 12, 24, 48. (3) $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$.

(2) 100, -50, 25, $-12\frac{1}{2}$. (4) a, ar, ar^2, ar^3, ar^4.

The first and last terms are often called the **Extremes** and the other terms the **Means.**

400. **The n th Term of a Geometrical Progression.** Let a represent the first term of any geometrical progression, and r the ratio. Then the progression may be written

No. of term: 1. 2. 3. 4. 5.

Progression: a, ar, ar^2, ar^3, ar^4, . . .

We notice that, by the nature of the progression, every time the number of terms is increased by 1 the exponent of r is increased by 1 also, and the exponent of r in any term is one less than the number of that term. Therefore, representing the n th term by l,

$$l=ar^{n-1}. \qquad [3]$$

401. **The Sum of n Terms of a Geometrical Progression.** Representing by s the sum of n terms of any geometrical progression, we have

$$s=a+ar+ar^2+ar^3+\ldots+ar^{n-2}+ar^{n-1}. \qquad (1)$$

Multiplying this equation by r, we get

$$rs=ar+ar^2+ar^3+ar^4+\ldots+ar^{n-1}+ar^n. \qquad (2)$$

Subtracting (1) from (2), we have

$$rs-s=ar^n-a. \qquad (3)$$

Whence $$s(r-1)=ar^n-a,$$

or $$s=\frac{ar^n-a}{r-1}. \qquad [4]$$

Now, from [3] $l=ar^{n-1}$. Therefore, $ar^n=r(ar^{n-1})=rl$, and [4] may be written

$$s=\frac{rl-a}{r-1}. \qquad [5]$$

402. We give a few examples of the use of formulas [3], [4], and [5].

(1) Find the 7th term of the progression $4+8+16+\ldots$

Here $a=4$, $r=2$, and $n=7$, whence

$$l=4\times 2^6=256.$$

(2) Find sum of 6 terms of progression $13+1.3+.13+\ldots$

Here $a=13$, $n=6$, and $r=\frac{1}{10}$, whence

$$s=\frac{13\times(\frac{1}{10})^6-13}{\frac{1}{10}-1},$$

that is, $$s=\frac{13-13000000}{100000-1000000}=\frac{12999987}{900000}=14.44443.$$

(3) Insert 3 geometrical means between 31 and 496.

Here $a=31$, $l=496$, and $n=5$, whence

$$496=31\times r^4,$$

or $$r^4=16,$$

therefore, $$r=\pm 2.$$

Consequently, the required means are 62, 124, and 248, or -62, $+124$, and -248.

403. The two equations

$$\begin{cases} l=ar^{n-1} \\ s=\dfrac{ar^n-a}{r-1} \end{cases}$$

contain five letters. If any two of them are unknown numbers and the values of the other three are given, the value of the two unknown numbers can be determined by solving the system of two equations. But if r is an unknown number, the equations of the system are of a high degree, since n is usually a large number and always greater than 2 at least. In this case we will be unable to solve the system, as it is beyond the range of Chapter XV. Also, if n is an unknown number, we will have an equation with the unknown number appearing as an *exponent*, which is a kind of equation we have not yet considered. Hence there are a limited number of cases in which, with our present means, we can solve the above system. We give a few examples of the cases readily solved.

(1) Find the sum of a geometrical progression of 7 terms, of which the last term is 128, the ratio being 2.

Here $l=128$, $r=2$, and $n=7$, whence

$$\begin{cases} 128=a\,2^6 & (1) \\ s=\dfrac{a\,2^7-a}{1} & (2) \end{cases}$$

From (1) $a=2$, whence, from (2), $s=254$.

(2) Find the sum of a geometrical progression of 5 terms, the extremes being 8 and 10368.

Here $a=8$, $l=10368$, and $n=5$, whence

$$\begin{cases} 10368=8r^4 & (1) \\ s=\dfrac{r\,10368-8}{r-1} & (2) \end{cases}$$

From (1) $r=6$, whence from (2) $s=12450$.

(3) Find the extremes of a geometrical progression whose sum is 635, if the ratio is 2 and the number of terms 7.

Here $s=635$, $r=2$, and $n=7$, whence

$$\begin{cases} l=a\,2^6 & (1) \\ 635=\dfrac{2l-a}{1} & (2) \end{cases}$$

Substituting l from (1) in (2), we get

$$635=128a-a.$$

Whence $a=5$; hence $l=320$.

(4) The 4th term of a geometrical progression is 4, and the 6th term is 1. What is the 10th term?

Here $ar^3=4$ (1)

and $ar^5=1$ (2)

Whence, by dividing (2) by (1),

$$r^2=\tfrac{1}{4}.$$

Whence, $r=\pm\tfrac{1}{2}$.

Therefore, from (1), $a=\dfrac{4}{r^3}=\pm 32$.

Then the 10th term is $\pm 32(\pm\tfrac{1}{2})^9=\tfrac{1}{16}$.

EXERCISE 140.

EXAMPLES AND PROBLEMS.

1. Find the sum of 7 terms of $4+8+16+\ldots$
2. Find the sum of 9 terms of $2+6+18+\ldots$
3. Find the sum of 7 terms of $1+4+16+\ldots$
4. Find the sum of 11 terms of $9+3+1+\ldots$
5. Find the sum of 10 terms of $1+\tfrac{1}{2}+\tfrac{1}{4}+\ldots$
6. Find the 10th term and the sum of 10 terms of $4-2+1-\ldots$
7. Sum the series $\sqrt[6]{3}+\sqrt[6]{6}+\sqrt[6]{12}+\ldots$ to 8 terms.
8. Sum the series $3-2+\tfrac{4}{3}-\tfrac{8}{9}+\ldots$ to 9 terms.
9. Sum the series $-4+8-16+32-\ldots$ to 6 terms.
10. Sum $a+a(1+x)+a(1+x)^2+\ldots$ to 8 terms.

11. Sum $a+\frac{a}{a^2-1}+\frac{a}{(a^2-1)^2}+\ldots$ to 10 terms.

12. Sum $b(1+x)^{n-1}+b(1+x)^{n-2}+\ldots$ to n terms.

13. Sum $x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3+\ldots$ to n terms.

14. Sum $x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+\ldots$ to n terms.

15. Find r and s; given $a=2$, $l=31250$, $n=7$.

16. Find r and s; given $a=36$, $l=\frac{4}{81}$, $n=7$.

17. Find r and s; given $a=3$, $l=49152$, $n=8$.

18. Find r and s; given $a=7$, $l=3584$, $n=10$.

19. Insert 2 geometrical means between 47 and 1269.

20. Insert 3 geometrical means between 2 and 3.

21. Insert 1 geometrical mean between 14 and 686.

22. Given $l=78125$, $r=5$, $n=8$; find a and s.

23. Given $l=\frac{2}{27}$, $r=\frac{1}{3}$, $n=5$; find a and s.

24. Given $s=635$, $n=7$, $r=2$; find a and l.

25. Select 6 terms from the progression $\frac{1}{2}-2+8-\ldots$ whose sum shall equal -6536.

26. The sum of the extremes of a geometrical progression of 4 terms is 56, and the sum of the means is 24. Find the 4 terms.

27. Insert 7 geometrical means between a^8 and b^8.

28. Given $a=\frac{1}{8}$, $l=1024$, $n=14$; find r and s.

29. Sum the series $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\frac{1}{2^4}+\frac{3}{2^5}+\frac{5}{2^6}+\ldots$ to 9 terms, or the progression $\left(\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}\right)+\left(\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}\right)\frac{1}{2^3}+$ to 3 terms.

30. Find the sum of the first 10 consecutive powers of 2.

31. Find sum of the first 10 consecutive powers of $-\frac{1}{2}$.

32. Sum the progression .272727 . . . or $\frac{27}{100}+\frac{27}{10000}+\frac{27}{1000000}+\ldots$ to 6 terms.

33. Sum $a-ar^{-1}+ar^{-2}-ar^{-3}+\ldots$ to n terms.

34. The 4th term of a geometrical progression is 192 and the 7th term is 12288; find the sum of the first 3 terms.

35. The 6th term of a geometrical progression is 156, and the 8th term is 7644; what is the 4th term?

36. Prove that if numbers are in geometrical progression their differences are also in geometrical progression, having the same common ratio as before.

37. If $a+b+c+d+\ldots$ is a geometrical progression, prove that $(a^2+b^2)+(b^2+c^2)+(c^2+d^2)+\ldots$ is also a geometrical progression.

38. A man agreed to pay for the shoeing of his horse as follows: .0001 cents for the first nail, .0002 cents for the second nail, .0004 cents for the third nail, and so on until the 8 nails in each shoe were paid for. How many dollars did he agree to pay? How much did the last nail cost him?

CHAPTER XX.

BINOMIAL THEOREM.

EXERCISE 141.

THE LAWS OF EXPONENTS AND COEFFICIENTS.

404. The **Binomial Theorem** enables us to find any power of a binomial without the labor of obtaining the previous powers; in other words, it enables us to obtain any power of a binomial without actually performing the multiplication.

405. Let us obtain several powers of $x+a$ by actual multiplication :

1st power, $x+a$

$x+a$

x^2+ax

$ax+a^2$

2d power, $x^2+2ax+a^2$

$x+a$

$x^3+2ax^2+a^2x$

$ax^2+2a^2x+x^3$

3d power, $x^3+3ax^2+3a^2x+x^3$

$x+a$

$x^4+3ax^3+3a^2x^2+a^3x$

$ax^3+3a^2x^2+3a^3x+a^4$

4th power, $x^4+4ax^3+6a^2x^2+4a^3x+a^4$

$x+a$

$x^5+4ax^4+6a^2x^3+4a^3x^2+a^4x$

$ax^4+4a^2x^3+6a^3x^2+4a^4x+a^5$

5th power, $x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$

When any power of $x+a$ is written out in full, as in the above, it is called the **Expansion** of that power of $x+a$.

406. The Law of Exponents. We notice in the expansion of the different powers of $x+a$ that x appears in each term except the last, and that a appears in each term except the first, and that both x and a occur in each of the other terms. We observe also that

	THE EXPONENTS OF x FOLLOW THE FOLLOWING SCHEME.	THE EXPONENTS OF a FOLLOW THE FOLLOWING SCHEME.
In $(x+a)^1$	1 0	0 1
In $(x+a)^2$	2 1 0	0 1 2
In $(x+a)^3$	3 2 1 0	0 1 2 3
In $(x+a)^4$	4 3 2 1 0	0 1 2 3 4
In $(x+a)^5$	5 4 3 2 1 0	0 1 2 3 4 5

It is a necessary consequence of the successive multiplication by $x+a$, that the exponents will continue to fall into the above schemes, for at each multiplication all the power of x, likewise of a, are increased by 1, and there will always be one term which does not contain x, and always one which does not contain a; therefore the law of exponents:

In any power of a binomial, $x+a$, the exponent of x begins in the first term with the exponent of the power, and in the following terms continually decreases by one. The exponent of a commences with one in the second term, and continually increases by one.

407. Number of Terms. We observe that in the *first* power there are *two* terms, in the *second* power there are *three* terms, in the *third* power there are *four* terms, and so on. In *any* power of $x+a$ there is a term containing each of the powers of x, as high as the required power of $x+a$, and one term which does not contain x; therefore,

The number of terms in any power of a binomial is always one greater than the exponent of the power.

408. The Law of Coefficients. First Statement. The coefficients* in any power of $x+a$ are of course the sums of the coefficients in the two partial products which are added together to produce the required power. Now the *coefficients* in each of the partial products are just alike, and each partial product has the same coefficients as the multiplicand, or next lower power of $x+a$ to the one we are finding. Thus the coefficients which are added together to produce the coefficients in $(x+a)^4$ are as follows:

$$\begin{array}{ccccc} 1 & 3 & 3 & 1 & \\ & 1 & 3 & 3 & 1 \\ \hline 1 & 4 & 6 & 4 & 1 \end{array}$$

The *second* coefficient in $(x+a)^4$ is the sum of the *second* and *first* coefficients in $(x+a)^3$, the *third* coefficient in $(x+a)^4$ is the sum of the *third* and *second* coefficients in $(x+a)^3$, the *fourth* coefficient in $(x+a)^4$ is the sum of the *fourth* and *third* coefficients in $(x+a)^3$, the *fifth* coefficient in $(x+a)^4$ is the sum of the *fifth* (which is 0) and *fourth* coefficients in $(x+a)^5$. Writing down the coefficients in the different powers of $x+a$, we may present this same truth in a little different form:

Coefficients in $(x+a)^1$ 1 1
Coefficients in $(x+a)^2$ 1 2 1
Coefficients in $(x+a)^3$ 1 3 3 1
Coefficients in $(x+a)^4$ 1 4 6 4 1
Coefficients in $(x+a)^5$ 1 5 10 10 5 1

By arranging the coefficients in this triangle we may say that *each coefficient is the sum of the coefficient immediately above it and the coefficient immediately to the left of this last.*

* By the coefficients in any power of $x+a$ is meant the *coefficients of the powers of x and a.* This language is not exact, (see Art. 14,) but it is customary.

Since the partial products will continue to be combined in the way we have noticed if we continue to multiply by $x+a$, we may extend the above triangle indefinitely and get the coefficients in any power of $x+a$ we wish:

Coefficients in $(x+a)^1$ 1 1
Coefficients in $(x+a)^2$ 1 2 1
Coefficients in $(x+a)^3$ 1 3 3 1
Coefficients in $(x+a)^4$ 1 4 6 4 1
Coefficients in $(x+a)^5$ 1 5 10 10 5 1
Coefficients in $(x+a)^6$ 1 6 15 20 15 6 1
Coefficients in $(x+a)^7$ 1 7 21 35 35 21 7 1
Coefficients in $(x+a)^8$ 1 8 28 56 70 56 28 8 1

409. The Law of Coefficients. Second Statement. It is more usual to use a law of coefficients different from the one given in the previous article. It is found that *the coefficient in any term in a power of a binomial can be made from the coefficient and the two exponents in the preceding term.* Thus consider the *fourth* power of $x+a$,

$$x^4+4ax^3+6a^2x^2+4a^3x+a^4.$$

The first coefficient is 1, and the second coefficient is 4, the exponent of the power; the third coefficient, 6, is the preceding coefficient, 4, multiplied by 3, the exponent of x in the second term, divided by 2, one more than the exponent of a in the second term. Also, 4, the fourth coefficient, is $\frac{6\times 2}{3}$, where 6 is the coefficient in the preceding term, 2 is the exponent of x, and 3 is one more than the exponent of a; and so on.

Likewise in the fifth power of $x+a$,

$$x^5+5ax^4+10a^2x^3+10a^3x^2+5ax^4+x^5,$$

the coefficient in the first term is 1; the coefficient in the second term is 5, the exponent of the required power; the coefficient in the third term, 10, is the coefficient in

the preceding term, 5, multiplied by 4, the exponent of x, and divided by 2, one more than the exponent of a; 10, the coefficient in the fourth term, is similarly $\frac{10\times 3}{3}$; 5, the next coefficient, is $\frac{10\times 2}{4}$; and 1, the last coefficient, is $\frac{5\times 1}{5}$. The next coefficient would be $\frac{1\times 0}{6}$.

After treating any other of the first five powers in the same way, we would find,

In any of the first five powers of $x+a$, the coefficient in the first term is 1, *that in the second term is the exponent of the power, and if the coefficient in any term be multiplied by the exponent of x in that term and divided by the exponent of a increased by one, it will give the coefficient in the succeeding term.*

410. The law stated in the last article may be observed to be true in any of the five powers of $x+a$ that we have actually worked out; it now remains to prove that the same law holds for *all* powers of $x+a$.

Let n stand for a positive whole number, and suppose we wish to find $(x+a)^n$. We know from Art. 406 that the terms *without* the coefficients will be as follows:

$$x^n,\ ax^{n-1},\ a^2x^{n-2},\ a^3x^{n-3},\ a^4x^{n-4},\ \text{etc.}$$

Now, if the law stated in the previous article *does* hold, we would write $(x+a)^n=$

$$x^n+nax^{n-1}+\frac{n(n-1)}{2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{2\times 3}a^3x^{n-3}$$
$$+\frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}a^4x^{n-4}+\ldots \quad (1)$$

If this is true, by Art. 408 the coefficients in $(x+a)^{n+1}$ are

$$1,\ n+1,\ \frac{n(n-1)}{2}+n,\ \frac{n(n-1)(n-2)}{2\times 3}+\frac{n(n-1)}{2},$$
$$\frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}+\frac{n(n-1)(n-2)}{2\times 3},\ \text{etc.}$$

or, removing a common factor from the third, fourth, etc., of these expressions, we would write them

$$1,\ n+1,\ n\left(\frac{n-1}{2}+1\right),\ n(n-1)\left(\frac{n-2}{2\times 3}+\frac{1}{2}\right),$$
$$n(n-1)(n-2)\left(\frac{n-3}{2\times 3\times 4}+\frac{1}{2\times 3}\right),\ \text{etc.}$$

or, performing the additions in the large parentheses,

$$1,\ n+1,\ \frac{(n+1)n}{2},\ \frac{(n+1)n(n-1)}{2\times 3},$$
$$\frac{(n+1)n(n-1)(n-2)}{2\times 3\times 4},\ \text{etc.}$$

or, supplying the powers of x and a for the $(n+1)$ power of $x+a$ by Art. 406, we get

$$(x+a)^{n+1}=$$
$$x^{n+1}+(n+1)ax^n+\frac{(n+1)n}{2}a^2x^{n-1}+\frac{(n+1)n(n-1)}{2\times 3}a^3x^{n-2}$$
$$+\frac{(n+1)n(n-1)(n-2)}{2\times 3\times 4}a^4x^{n-3}+\ldots \qquad (2)$$

Now we know that equation (2) is true *if equation* (1) *is true.* But equation (2) is of exactly the same form as (1), merely having $(n+1)$ in place of n, each coefficient being obtainable from the coefficient and exponents of the preceding term by the law of Art 409. Therefore we have proved that *the law of coefficients of Art.* 409 *holds in the* $(n+1)$ *power of* $x+a$ *if it holds in the* n*th power of* $x+a$. But we *know* this law of coefficients holds in the 5th power of $x+a$; therefore it follows that it holds in the 6th power. Now we know this law of coefficients holds in the 6th power of $x+a$, and therefore it holds in the 7th power; therefore it holds in the 8th power, and so on. Therefore it holds universally.

Thus we have proven the law of Art. 409 holds for any power of a binonial.

412. The statement of the laws of exponents and coefficients for any power of $x+a$ is called the **Binomial Theorem**, and is usually given as follows:

I. EXPONENTS. *In any power of a binomial, $x+a$, the exponent of x begins in the first term with the exponent of the power, and in the following terms continually decreases by one. The exponent of a commences with one in the second term of the power, and continually increases by one;*

II. COEFFICIENTS. *The coefficient in the first term is one, that in the second term is the exponent of the power; and if the coefficient in any term be multiplied by the exponent of x in that term and divided by the exponent of a increased by one, it will give the coefficient in the succeeding term.*

413. The expansion of $(x\pm a)^n$ is usually called the **Binomial Formula.**

If in equation (1), Art. 400, we substitute $\pm a$ for a, we get the following as the expansion of $(x\pm a)^n$:

$$(x\pm a)^n =$$

$$x^n \pm nax^{n-1} + \frac{n(n-1)}{2}a^2x^{n-2} \pm \frac{n(n-1)(n-2)}{2\times 3}a^3x^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2\times 3\times 4}a^4x^{n-4} \pm \ldots \quad [1]$$

Therefore, in any power of the *difference* of two numbers the sign of the first term is $+$, of the second $-$, and so on, alternately $+$ and $-$.

414. We will now give a few examples of the use of the binomial theorem.

(1) Expand $(a+b)^6$.

We may expand this at once by the theorem as follows:

$$a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6,$$

or we may substitute $x=a$, $a=b$, and $n=6$ in formula [1]

$$a^6+6a^5b+\frac{6\times5}{2}a^4b^2+\frac{6\times5\times4}{2\times3}a^3b^3+\frac{6\times5\times4\times3}{2\times3\times4}a^2b^4$$
$$+\frac{6\times5\times4\times3\times2}{2\times3\times4\times5}ab^5+\frac{6\times5\times4\times3\times2\times1}{2\times3\times4\times5\times6}b^6,$$

which reduces to the same result as before.

(2) Expand $(u+3y)^5$.

Here $x=u$ and $a=3y$. By the theorem we get

$$u^5+5u^4(3y)+10u^3(3y)^2+10u^2(3y)^3+5u(3y)^4+(3y)^5.$$

Performing the indicated operations, we get

$$u^5+15u^4y+90u^3y^2+270u^2y^3+405uy^4+243y^5.$$

(3) Expand $(r^2-2)^4$.

Here $x=r^2$, $a=-2$, and $n=4$. By the theorem

$$(r^2)^4-4(r^2)^32+6(r^2)^22^2-4(r^2)2^3+2^4.$$

Performing the indicated operations, we get

$$r^8-8r^6+24r^4-32r^2+16.$$

(4) Expand $(2b-\frac{1}{2})^3$.

Here $x=3b$, $a=-\frac{1}{2}$, $n=3$. By the theorem,

$$(3b)^3-3(3b)^2(\tfrac{1}{2})+3(3b)(\tfrac{1}{2})^2-(\tfrac{1}{2})^3.$$

Performing the indicated operations, we get

$$27b^3-\tfrac{27}{2}b^2+\tfrac{9}{4}b-\tfrac{1}{8}.$$

EXERCISE 142.

EXAMPLES.

Expand each of the following by the binomial theorem or formula:

1. $(a+x)^3$.
2. $(b+x)^4$.
3. $(d+y)^5$.
4. $(c+x)^6$.
5. $(1+a)^{10}$.
6. $(2+x)^2$.
7. $(2-x)^4$.
8. $(\frac{1}{2}+x)^5$.
9. $(m+u)^9$.
10. $(a-x)^7$.
11. $(5d-3y)^5$.
12. $(a^3-b^2)^8$.

13. $(-x+2a)^7$.
14. $(2x+3a)^6$.
15. $(1-x)^6$.
16. $(1-a)^3$.
17. $(b^2-c^2)^5$.
18. $(x+2c)^4$.
19. $(3a+\frac{1}{2})^6$.
20. $(2ax-x^2)^4$.
21. $(2h^2-3x^3)^5$.
22. $(3x^2-1)^4$.
23. $(\sqrt{a}+x)^6$.
24. $(2b-\sqrt{x})^5$
25. $(xy-x^2)^6$.
26. $(\sqrt{ab}-\sqrt[3]{ab})^6$.
27. $(a+[x+y])^3$.
28. $([a+b]-2)^3$.
29. $(a+b-y)^3$.
30. $([a+b]+[c+d])^3$
31. $(a-b+x-2)^3$.
32. $(x^{\frac{2}{3}}+x^{\frac{3}{2}})^6$.
33. $(a^{-2}-b^{\frac{1}{2}})^4$.
34. $(x^2+2ax+a^2)^3$.
35. $(\sqrt{c^2-2x})^{12}$.
36. $\left(x+\frac{1}{x^2}\right)^8$.

THE END

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